Master Exam-2005 Fall
Basic Areas

Do six of the following problems. But you must do at least two problems in real variable, one problem in abstract algebra, one problem in topology, and one problem in linear algebra.

1. (Linear algebra) Let V be a finite dimensional vector space and S, T two linear operators from V to V. Prove that $\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$.

2. (Linear algebra) Suppose $(v_1, \ldots, v_n)$ is linearly independent in a vector space V and $w \in V$. Show that if $(v_1 + w, v_2 + w, \ldots, v_n + w)$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_n)$.

3. (Algebra) Let G be a group. Prove the following:

   (a) Its identity element is unique. (In this proof, you cannot use any inverse of $a \in G$.)

   (b) For any $a \in G$, the inverse of a is unique.

   (c) Let a^{-1} denote the inverse of a. For any $a, b \in G$, $(ab)^{-1} = b^{-1}a^{-1}$.

4. (Algebra) Let G be a group and a an element in G. Show that $C(a) = \{g \in G : ga = ag\}$ is a subgroup of G.

5. (Topology) Show that the continuous image of a compact set is compact.

6. (Topology) Let (M, d) be a metric space. Show that if (M, d) is Lindelöf, then (M, d) is separable. Also give an example of a topological space (X, τ) that is Lindelöf but not separable.

7. (Topology or Real Variable) Prove that if a function f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a, b]$.

8. (Real Variable) Let the sequence (x_n) be defined as $x_1 = 3$ and

   $$x_{n+1} = \frac{1}{4 - x_n}$$

   for all $n > 1$. Show that (x_n) is convergent and find its limit.

9. (Real Variable) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

10. (Real Variable) Let (a_n) and (b_n) be two sequences of real numbers. Prove:

    (a) If $\sum_{k=1}^{\infty} a_k$ exists, then $\lim_{n\to\infty} a_n = 0$.

    (b) If both (a_n) and (b_n) are convergent sequences, then $(a_n + b_n)$ is a convergent sequence.