Master Exam-2006 Spring
Basic Areas

Do six of the following problems. But you must do at least one problem in real variable, one problem in abstract algebra, one problem in topology, and one problem in linear algebra.

1. (Linear algebra) Let V be the set of all 3×3 hermitian matrices over $\mathbb{C}$. What is $\dim_{\mathbb{R}} V$ (the dimension of V over $\mathbb{R}$)? Find a basis of V over $\mathbb{R}$.

2. (Linear algebra) A set $\{e_1, e_2, \ldots, e_n\}$ of an inner product space V over $\mathbb{C}$ is called orthonormal if $\|e_i\| = 1$ for all $i \leq n$, and $\langle e_i, e_j \rangle = 0$ for all $i < j \leq n$. Show that every othonormal set is linearly independent.

3. (Abstract algebra) State and prove Lagrange's theorem.

4. (Abstract algebra) Let R be an integral domain with identity. Show that R is a field if $\{0\}$ is a maximal ideal of R.

5. (Abstract algebra) Let N be a normal subspace of G and H a subgroup of G. Show that NH is a subgroup of G.

6. (Topology) The lower limit topology τ of $\mathbb{R}$ is the topology generated by $\{[a, b) : a < b\}$. Is $[0, 1]$ compact in the lower limit topology? Prove your argument.

7. (Topology) Let (M, d) be a compact metric space and f a continuous function from M to $\mathbb{R}$. Show that f is uniformly continuous.

8. (Real Variable) Let f be a function from $[0, 1]$ to $\mathbb{R}$. Show that if f is differentiable at $x = \frac{1}{2}$, then f is continuous at $x = \frac{1}{2}$.

9. (Real Variable) Show that if a function is continuous on all of $\mathbb{R}$ and equal to 0 at every rational point, then it must be identically 0 on all of $\mathbb{R}$.

10. (Real Variable) Let $f_n(x) = \frac{x^2+nx}{n}$ and $f(x) = x$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Show the following:

    (a) $\lim_n f_n(x) = f(x)$ for every $x \in \mathbb{R}$.

    (b) $f_n(x)$ does not converge to $f(x)$ uniformly on $\mathbb{R}$.

    (c) $f_n(x)$ converges to $f(x)$ uniformly on every interval $[-b, b]$, $b > 0$.