Masters' Comprehensive Exam: Core Topics

September 13, 2008

Answer any **four** *of the following six problems.*

1. A function f satisfies a *Lipschitz condition* at $x \in (a, b)$ if there is some $M > 0$ and $\varepsilon > 0$ so that for all $y \in (a, b)$ with $|x - y| < \varepsilon$ we have $|f(x) - f(y)| \leq M|x - y|$.
   (a) Show also that if f is differentiable at x, then f satisfies a Lipschitz condition there.
   (b) Give an example of a function which satisfies a Lipschitz condition at a point, but which is not differentiable there.
   (c) Give an example of a function which fails to satisfy a Lipschitz condition at a point of continuity.

2. Define $\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} - \int_1^n \frac{1}{t}\, dt$. Prove that $\{\gamma_n\}_{n=0}^{\infty}$ converges (without using properties of the logarithm — this is a Riemann integration problem).

3. Show that if $f \colon \mathbb{R} \to \mathbb{R}$ satisfies the open set definition of continuity then it satisfies the ε-δ condition for continuity.

4. Let $Z(G) = \{x \in G \mid xy = yx \text{ for all } y \in G\}$ be the center of the group G.
   (a) Show that $Z(G)$ is a normal subgroup of G.
   (b) Show that if $G/Z(G)$ is cyclic then G is abelian.
   (c) Deduce that if G is non-abelian, then there exists an abelian subgroup H of G with $Z(G) \subsetneq H \subsetneq G$.

5. Let G be the group of $n \times n$ invertible real matrices under matrix multiplication and let H be those $n \times n$ matrices which have determinant 1.
   (a) Show that H is a normal subgroup of G.
   (b) Identify up to isomorphism the quotient group G/H.

6. Suppose $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ are two bases of a vector space V. Show that $n = m$.