Masters' Comprehensive Exam: Core Topics

February 9, 2008

Answer any **five** *of the following seven problems.*

1. Define a sequence $(x_n)_{n=0}^{\infty}$ by $x_0 = 2$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for $n \geq 0$.
   (a) Show that x_n converges as $n \to \infty$.
   (b) Identify the limit of the sequence $(x_n)_{n=0}^{\infty}$

2. Prove or disprove:
   (a) The product of two uniformly continuous functions on $\mathbb{R}$ is also uniformly continuous.
   (b) The product of two uniformly continuous functions on $[0, 1]$ is also uniformly continuous.

3. Let $(x_n)_{n=0}^{\infty}$ be a sequence of real numbers. Prove that the following are equivalent.
   (a) $\lim_{n\to\infty} x_n = a$.
   (b) Every subsequence of $(x_n)_{n=0}^{\infty}$ contains a subsequence that converges to a.

4. Prove that a compact subset of a Hausdorff space is closed.

5. Determine which of the following groups are isomorphic.
   (i) $\mathbb{Z}_2 \times \mathbb{Z}_4$, (ii) $\mathbb{Z}_8$, (iii) $\mathbb{Z}_2 \times \mathbb{Z}_3$, (iv) $\mathbb{Z}_6$.

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6. Let G be the group of 2×2 invertible upper triangular real matrices under matrix multiplication, i.e., matrices of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with $a, b, d \in \mathbb{R}$, $ad \neq 0$.

   (a) Show that the set K of matrices in G that are of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ form a normal subgroup of G.

   (b) Show that the quotient group G/K is isomorphic to $\mathbb{R}^\times \times \mathbb{R}^\times$.

7. Let $T \colon V \to V$ be a linear map on a real vector space V such that $T^2 = 1$.

   (a) Show that $V_+ = \{x \mid Tx = x\}$ and $V_- = \{x \mid Tx = -x\}$ are subspaces of V .

   (b) Show that V is isomorphic to the direct sum of V_+ and V_-.