Master Exam Part I-Fall 2010

Do six of the following eight problems. But you need to do at least one of each of the following area Algebra (1-2), Real Analysis (3-5), and Topology (6-7).

1. (Abstract algebra) If G is an abelian group and if $H = \{a \in G : a^2 = e\}$ (e is the identity of G), show that H is a subgroup of G.

2. (Abstract Algebra) If I, J are ideals of a ring R, define $I + J$ by

$$I + J = \{i + j : i \in I,\ j \in J\}.$$

   Prove that $I + J$ is an ideal of R.

3. (Real Analysis) Let f, g be two bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $f \cdot g$ is also continuous.

4. (Real Analysis) If $\sum a_n$ converges absolutely and $\sum b_n$ is a rearrangement of $\sum a_n$, show that $\sum a_n = \sum b_n$.

5. (Real Analysis) Find the radius and interval of convergence for each of the following power series.

   (a) $\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2}$.

   (b) $\sum_{n=1}^{\infty} \frac{(x-4)^n}{n}$.

6. (Topology) Let X be a topological space and let $\Delta = \{(x, x) : x \in X\}$. Show that X is Hausdorff if and only if Δ is a closed subset of $X \times X$ (in the product topology).

7. (Topology) Let (X, d) be a metric space. Recall that a subset O of X is open if for $x \in O$, O contains an open ball with center at x. Let x be any point in X and a a positive real. Show that the set $\{y : d(x, y) < a\}$ is open.

8. (Linear Algebra) Prove that $(\sum_{j=1}^{n} a_j b_j)^2 \leq (\sum_{j=1}^{n} j a_j^2)(\sum_{j=1}^{n} \frac{b_j^2}{j})$.