Master Exam Part I-Spring 2010

Do six of the following eight problems. But you need to do at least one of each of the following area Real Analysis (1-3), Algebra (4-5), and Topology (6-7).

1. Use the δ-ϵ proof to show that the product of two continuous functions is continuous.
2. (a) Give an example of a sequence (f_n) of continuous functions that converges to f pointwise that is not continuous.

   (b) Show that if (f_n) is a sequence of continuous functions that converges to f uniformly. Then f is continuous.
3. Let f be a function from $[a, b]$ to $\mathbb{R}$ such that f is differentiable at $c \in (a, b)$. Show that f is continuous at c.
4. Let M, N be two normal subgroups of G. Suppose that $M \cap N = \{e\}$. Show that $mn = nm$ for all $n \in N$ and $m \in M$.
5. Let $F \subseteq K \subseteq L$. Suppose that the dimension of L over K is $m(< \infty)$ and the dimension of K over F is $n(< \infty)$. What is the dimension of L over F? Prove your argument.
6. Let K be a compact set and f a continuous from K to $\mathbb{R}$. Show that there is $x \in K$ such that
   $$f(x) = \sup\{f(y) : y \in K\}.$$
7. Show that every metric space is first countable. Give an example of metric space that is not second countable.
8. Let T be a linear transformation from a vector space X to another vector space Y. Show that $\ker(T)$ is a subspace of X.