MASTER EXAM-PART I

Do six of the following eight problems; but you must do at least two problems from Real Variable, one problem from Abstract Algebra, and one problem from Topology.

1. (Real Variable) Show that the function

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   is differentiable on $\mathbb{R}$. Is f' continuous on $\mathbb{R}$?

2. (Real analysis) Let $f : [a, b] \to \mathbb{R}$ be a bounded function on a bounded interval.

   (a) Define the Riemann integral of f over the interval $[a, b]$, this is $\int_a^b f(x)dx$.

   (b) Suppose that f is a Riemann integrable function on a bounded interval, $[a, b]$. Use the definition of Riemann integration to show that $g(x) = f(-x)$ is Riemann integrable on the interval $[-b, -a]$ and

$$\int_a^b f(x)dx = \int_{-b}^{-a} f(-x)dx.$$

3. (Real Variable) Let $F(0, 1)$ be the set of all functions from $(0, 1)$ to $\mathbb{R}$ and let $f, f_n \in F(0, 1)$.

   (a) State the definition of uniform convergence of (f_n) to f.

   (b) Show that if (f_n) is a sequence of continuous functions that converges to f uniformly on $(0, 1)$, then f is continuous on $(0, 1)$.

4. (Abstract Algebra) Let G, G' be two groups and N a subgroup of G.

   (a) State the definition that N is a normal subgroup of G.

   (b) State the definition of homomorphism (from G to G').

   (c) Let ϕ be a homomorphism from G to G'. Show that $\ker(\phi) = \{g \in G : \phi(g) = e'\}$ (e' is the identity of G') is a normal subgroup of G.

5. (Abstract Algebra) Let R be a commutative ring with unit.

   (a) State the definitions of ideal and maximal ideal of R.

   (b) Let M be a proper ideal of R. Show that M is maximal ideal if and only if R/M is a field.

6. (Topology) Let (X, d) be a complete metric space and A a subset of X. Let ρ be a metric on A defined by

$$\rho(a, b) = d(a, b) \quad \text{for any } a, b \in A.$$

   Show that (A, ρ) is complete if and only if A is closed.

7. (Topology) Show that the continuous image of compact set is compact.

8. (Linear Algebra) Let T be a linear transformation from $\mathbb{C}^6$ to $\mathbb{C}^6$ with the minimal polynomial

$$f(\lambda) = (\lambda - 3)^2(\lambda - 1)^2$$

and the characteristic polynomial

$$f(\lambda) = (\lambda - 3)^4(\lambda - 1)^2$$

Write all possible Jordan normal forms.