Master Exam-Part I

1. (Probability) Suppose that two player A and B take turns tossing THREE fair coins, and the winner is the first player who obtains all THREE heads on a given toss of the THREE coins. Suppose that A started first. Find the probability that A will win.

2. (Probability) Bowl C contains six red chips and four blue chips. Three of these 10 chips are selected at random and without replacement and put in bowl D, which was originally empty. Two chips are then drawn at random from bowl D.

   (a) Let X be the number of blue chips transferred from bowl C to bowl D. What is the p.m.f. of X ? [i.e. find $f(x) = P(X = x)$.]

   (b) Find the mean of X.

   (c) Find the variance of X.

3. (Real Analysis) Use the ϵ-δ definition of continuity to show the function $f(x) = x^2$ is continuous.

4. (Real Analysis) Show that if f is differentiable, then f is continuous.

5. (Real Analysis) Let f be a continuous function on a bounded interval $[a, b]$. Show that f is Riemann integrable.

6. (Linear Algebra) Let T be a Hermitian operator from a finite dimensional Hilbert space H (over complex field) into itself. (Do two of the following three problems)

   (a) Show that every eigenvalue is real.

   (b) Show that if λ_1, λ_2 are two distinct eigenvalues of T and x_1, x_2 are two vectors such that $T(x_1) = \lambda_1 x_1$ and $T(x_2) = \lambda_2 x_2$. Show that $\langle x_1, x_2 \rangle = 0$.

   (c) Let X be a subspace of H such that $T(X) \subseteq X$. Show that if

   $$X^{\perp} = \{y : \langle y, x \rangle \text{ for all } x \in X\},$$

   then $T(X^{\perp}) \subseteq X^{\perp}$.

7. (Graph Theory) Prove that either the graph or its complement is connected.

8. (Graph Theory) Let T be a tree of order $n > 1$ with no vertices of degree 2. Prove that T has at least $\frac{n}{2} + 1$ leaves.