Statistics Masters Comprehensive Exam

November 2, 2002

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

1	2	3	4	5	6	7	8	9	10	11	12

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Let X_1 and X_2 be independently and identically distributed normal variables with mean 0 and variance 1. Put $Y_1 = a_1X_1 + a_2X_2$ and $Y_2 = b_1X_1 + b_2X_2$.

   (a) Obtain a set of constants $\{a_i, b_i, i = 1, 2\}$ for which Y_1 and Y_2 are independently distributed of each other.

   (b) Suppose that Y_1 and Y_2 are uncorrelated with each other and suppose that $a_1^2 + a_2^2 = b_1^2 + b_2^2$. Obtain the pdf (probability density function) of $Z = \frac{Y_1^2}{Y_1^2+Y_2^2}$. (Be sure to give the support of Z.)

2. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x;\theta) = e^{-\theta}\theta^x/x!, x = 0, 1, \ldots, \infty$. Put $\phi(\theta) = \theta^2 e^{-\theta}/2$.

   (a) Derive the MLE (Maximum Likelihood Estimator) of $\phi(\theta)$.

   (b) Derive the UMVUE (Uniformly Minimum Varianced Unbiased Estimator) of $\phi(\theta)$.

3. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x, \theta) = \theta^2 x^{\theta^2 - 1}, 0 < x < 1; f(x, \theta) = 0$, if $x \leq 0$ or $x \geq 1$.

   (a) Derive the size α UMP (Uniformly Most Powerful) test for testing $H_0 : \theta = 1$ vs $H_1 : \theta > 1$.

   (b) Derive the power function of your test.

   (c) Does the size α UMP test for testing $H_0 : \theta = 1$ vs $H_2 : \theta \neq 1$ exist ? Why ?

4. Let X and Y be jointly distributed random variables. If $X|Y = y \sim Poisson(y)$, and $Y \sim Exponential(\lambda)$

   (a) Find the marginal distribution of X

   (b) Find $P(Y > a|X = 1)$

5. It is hypothesized that in Collierville Tennessee, 10% of all households have 3PC's, 20% have 2PC's, 40% have 1PC, and 30% have no PC's. In a survey of the households in Collierville, a sample of 100 households is randomly selected. Assuming that the hypothesis is true, what is the probability that the **average number** of PC's among the selected households is less than 1.5?

6. Let $X_1, \ldots, X_n$ be a random sample from a Uniform $(\theta, 2)$ distribution.

   (a) find the maximum likelihood estimator of θ.

   (b) Is the MLE of θ a sufficient statistic for θ?

   (c) If $n = 400$ and $\theta = 1$, find an approximate value of a such that $P(\sum_{i=1}^{n} X_i > a) = 0.2$.

7. For a class of 30 students, the homework scores x_i's and the final scores y_i's were summarized as follows: $\sum_{i=1}^{30} x_i = 323$, $\sum_{i=1}^{30} y_i = 2282$, $\sum_{i=1}^{30} x_i^2 = 3650$, $\sum_{i=1}^{30} y_i^2 = 171460$ and $\sum_{i=1}^{30} x_i y_i = 24466$.

   (a) Find the estimated regression line of Y on X: $y = \hat{\alpha} + \hat{\beta}x$.

   (b) If a student in a similar situation in the future obtained a homework score of 11, (i) predict his final score and (ii) illustrate how to obtain a range of his final score with 95% confidence level.

8. Suppose that $\{X_1, X_2\}$ is a random sample from the normal distribution $N(0, \sigma^2)$.

(a) Let $Y = a_1X_1 + a_2X_2$ with constants a_1, a_2. (i) Give the distribution of Y and (ii) choose a_1, a_2 such that $Y \sim N(0, 1)$.

(b) (i) Show both $Z_1 = \frac{1}{2}\{X_1^2 + X_2^2\}$ and $Z_2 = X_1^2 + X_2$ are unbiased estimators for σ^2. (ii) Which of the two estimators do you recommend to use? Explain.

9. Let $Y_i \sim N(\beta x_i, \sigma^2)$ independently for $i = 1, \ldots, n$, where x_i's are the values of a non-random covariate.

   (a) Derive the maximum likelihood estimators for β and σ^2.

   (b) Let $\hat{\beta}$ be the maximum likelihood estimator of β, respectively. Show that

$$\sum_{i=1}^{n}(Y_i - \hat{\beta}x_i)x_i = 0.$$

10. Let X_1 and X_2 be independently and identically distributed from a geometric distribution with probability mass function given by

$$P(X = x) = p(1-p)^{x-1}, \text{ where } x = 1, 2, 3, \ldots.$$

(a) Find the UMVUE for p.

(b) Find the UMVUE for p^2.

11. Let X and Y be two random variables with joint density

$$f(x,y) = \begin{cases} 1, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Define $U = X + Y, V = X/(X+Y)$.

(a) Find the the joint density of U and V.

(b) Find the the marginal density of U.

(c) Find the the marginal density of V.

12. Let $X_1, \ldots, X_n$ be a random sample of size n from a population with density

$$f(x|\theta) = \theta(1+x)^{-(\theta+1)}, \quad x > 0, \theta > 0.$$

Suppose that θ has been drawn according to a prior distribution which is exponential with a known mean μ.

(a) Find the posterior distribution of θ.

(b) Find the the Bayes estimator of θ under the squared error loss function.