Statistics Masters Comprehensive Exam

March 21, 2003

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

1	2	3	4	5	6	7	8	9	10	11	12

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x, \theta) = \theta e^{-\theta\, x} \quad x > 0, \theta > 0$.

   (a) Derive the probability density function of the minimum $X_{(1)} = \min(X_1, \ldots, X_n)$.

   (b) Derive the probability density function of the maximum $X_{(n)} = \max(X_1, \ldots, X_n)$.

2. Let $\{X_1, \ldots, X_n\}$ be a random sample from the population with density $f(x, \theta) = \theta\, e^{-\theta x}, x > 0, \theta > 0$.

   Derive the UMP (Uniformly Most Powerful) size α test for testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$.

3. Let X be a discrete random variable with probability density function

$$f(x) = \frac{e^\theta}{(1+e^\theta)^{x+1}}, \quad x = 0, 1, \ldots,$$

where θ is a real parameter.

(a) Find $\Pr(X \text{ is odd})$.

(b) Suppose that X and Y are independent, each with probability function $f(x)$. Find $\Pr(X = Y)$.

(c) Let X and Y be as defined in (b). Find the probability function of $Z = X + Y$.

4. Suppose that $X_1, \ldots, X_n$ are independently and identically distributed with a density function

$$f(x|\mu) = \begin{cases} e^{-(x-\mu)}, & x \geq \mu \\ 0, & \text{otherwise.} \end{cases}$$

We are interested in the testing problem

$$H_0 : \mu = 0 \text{ versus } H_1 : \mu = 1.$$

Consider the critical region given by $\min(X_1, \ldots, X_n) > C$.

(a) For fixed n, find C so that type I error is 0.1.

(b) For fixed n, find C so that type II error is 0.1.

(c) Find a set of values for C and n so that both types of errors do not exceed 0.1.

5. Suppose that a medical study involved independent subjects $1, 2, \ldots, 10$. The lifetimes of subjects $1, 2, \ldots, 7$ were observed as $6, 10, 13, 16, 18, 22, 23$ (in years), respectively. Information available showed that subjects $8, 9, 10$ survived beyond $7, 16, 20$ (in years), respectively. Consider that a subject's lifetime T follows the piecewise exponential model below.

$$T \sim f(t; \lambda_1, \lambda_2) = \begin{cases} \lambda_1 e^{-\lambda_1 t} & 0 \leq t < 10 \\ \lambda_2 e^{10(\lambda_2 - \lambda_1) - \lambda_2 t} & 10 \leq t < \infty \end{cases}$$

(a) Write down the likelihood function of λ_1 and λ_2.

(b) Derive the maximum likelihood estimates of λ_1 and λ_2.

6. An automobile manufacturer wished to study gasoline consumption. Five cars of the same model with manual transmission were randomly selected from the assembly line, and each of the cars was driven by eight drivers over a 40-miles test course and the miles per gallon were recorded. The fixed effect analysis of variance (ANOVA) model in a single study was first considered.

   (a) Write down the ANOVA model and interpret the model parameters with reference to the problem.

   (b) Complete the following ANOVA table.

source	sum squares	degree of freedom	mean sum squares
A (car)	94.71		
error			
total	380.96		

   (c) Test whether there is any significant difference among cars on gasoline consumption (use $\alpha = 0.05$). $F_{0.05}(4, 30) = 2.69$, $F_{0.05}(4, 40) = 2.61$, $F_{0.05}(5, 30) = 2.53$, and $F_{0.05}(7, 40) = 2.25$.

7. University officials are planning to audit 1586 new appointments to estimate the proportion p who have been incorrectly processed by the Payroll Department.

   (a) How large does the sample size need to be in order for the sample proportion, to have an 85% chance of lying within 0.03 of p? (i.e., the difference between the sample proportion and the true p is not larger than 0.03)

   (b) Past audits suggest that p will not be larger than 0.10. Using that information, recalculate the sample size asked for in Part (a).

8. Define $X \sim Gamma(\alpha, \lambda)$ if X has density

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, x > 0.$$

Let $X_1, \ldots, X_n$ be a random sample from a population with unknown parameter θ.

(a) Suppose that the population has a Gamma$(1, e^\theta)$ distribution.

i. Find the maximum likelihood estimator of θ.

ii. Find the MLE of the median of the population distribution.

(b) If the population has a Gamma (β, θ) distribution find the method of moments estimator of β and θ.

9. Let $X_1, \ldots, X_4$ be a random sample an Normal$(\theta, 1)$ population. We wish to test $H_0 : \theta = 2$ versus $H_1 : \theta = 1$.

   (a) Consider the following tests procedures

      i. Reject H_0 if $\sum_{i=1}^{4} X_i < C_1$.

      ii. Reject H_0 if $\min\{X_1, \ldots, X_4\} < C_2$.

      Find C_1 and C_2 so that the probability of Type I error is .05

   (b) Calculate the power of the tests and compare the two tests.

10. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with mean θ. Assume that θ is a random variable with a prior distribution

$$\pi(\theta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \theta^{\alpha-1} e^{-\theta/\beta}, \theta \geq 0, \alpha > 0, \beta > 0,$$

where α and β are some known fixed constants.

(a) Using the square loss function

$$L(\theta, a) = (\theta - a)^2,$$

find the Bayes estimator of θ.

(b) Using the loss function

$$L(\theta, a) = e^{\beta\theta}(\theta - a)^2,$$

find the Bayes estimator of θ.

11. Suppose that $X_1, \ldots, X_n$ form a random sample from a Poisson distribution with unknown mean θ. Let $g(\theta) = P(X_1 + X_2 = 1)$.

   (a) Find an unbiased estimator of $g(\theta)$.

   (b) Find a uniform minimum variance unbiased estimator (UMVUE) T_n of $g(\theta)$.

12. Let X_1, X_2 be two independent random variables with common probability density function

$$f(x) = \begin{cases} xe^{-x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

(a) Find the joint probability density functions of random variables $Y_1 = X_1 + X_2$ and $Y_2 = \dfrac{X_1}{X_1 + X_2}$.

(b) Find the marginal probability density functions of Y_1 and Y_2.