Statistics Masters Comprehensive Exam

November 13, 2004

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

1	2	3	4	5	6	7	8	9	10	11	12

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Let $\{X_1, \ldots, X_n\}$ be independently and identically distributed normal variables with mean 0 and variance 1. Put $Y_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $Y_2 = \sum_{i=1}^{n}(X_i - Y_1)^2$.

   (a) Show that Y_1 and Y_2 are independently distributed of each other.

   (b) What is the sampling distribution of Y_2?

2. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x;\theta) = \frac{1}{\theta_2 - \theta_1}, \theta_1 < x < \theta_2$.

(a) Derive the MLE (Maximum Likelihood Estimator) $\hat{\theta}_1$ and $\hat{\theta}_2$ of θ_1 and θ_2 respectively.

(b) Show that the estimators from (a) form a set of sufficient and complete statistics for $\{\theta_1, \theta_2\}$?

3. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x, \theta) = \theta^{-1} e^{-x/\theta}, 0 < x; f(x, \theta) = 0$, if $x \leq 0$.

   (a) Derive the size α UMP (Uniformly Most Powerful) test for testing $H_0 : \theta = 1$ vs $H_1 : \theta > 1$.

   (b) Derive the power function of your test.

   (c) Based on observed data $\{x_1, \ldots, x_n\}$, obtain the p- value of your test.

4. Let $\{X_1, \ldots, X_n\}$ be a random sample from the normal density with mean μ_1 and variance σ^2. Let $\{Y_1, \ldots, Y_m\}$ be a random sample from the normal density with mean μ_2 and variance $4\sigma^2$.

   (a) Derive the size α likelihood ratio test for testing $H_0 : \mu_1 = \mu_2$ vs $H_1 : \mu_1 \neq \mu_2$ when σ^2 is unknown.

   (b) Derive the sampling distribution of the test statistic under H_0.

5. Let X and Y be jointly distributed random variables. If $X|Y = y \sim Poisson(y)$, and $Y \sim Exponential(\lambda)$, where $E(Y) = 1/\lambda$.

   (a) Find the marginal distribution of X.

   (b) Find $P(Y > a|X \leq 1)$, where a is a constant.

6. Let $(X_1, X_2, X_3) \sim \text{Trinomial}(n, p_1, p_2, p_3)$, that is,

$$P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3},$$

where $p_3 = 1 - p_1 - p_2,\ X_3 = n - X_1 - X_2$, find $P(X_1 = k | X_1 + X_2 = m)$.

7. Consider a population with the following distribution: random variable with

x	0	2	4	6
P(X=x)	0.1	0.4	0.2	0.3

You take a random sample of size 400 from the population. Let S denote the sample sum. Find b such that $P(S > b) = 0.975$.

8. Let $X_1, \ldots, X_n$ be a random sample from a Uniform $(2, \theta)$ distribution.

(a) Find the maximum likelihood estimator of θ.

(b) Is the MLE of θ a sufficient statistic for θ?

(c) Is the MLE an unbiased estimator of θ?

9. Let the joint distribution of X and Y be given as

$$f(x, y) = 2e^{-(x+y)}, \quad 0 < x < y < \infty.$$

(a) Find the joint p.d.f. of X and $X + Y$.

(b) Find the marginal p.d.f.s of X and $X + Y$.

10. Let $X_1, X_2, \cdots, X_n$ be a random sample from the Bernouli distribution, say $P(X = 1) = \theta$ and $P(X = 0) = 1 - \theta$. We are interested in estimating $g(\theta) = \theta(1 - \theta)$.

    (a) Find the Cramer-Rao lower bound for the variance of unbiased estimators of $g(\theta)$.

    (b) Find the UMVUE of $g(\theta)$, if such exists.

11. Let $X_1, X_2, \cdots, X_n$ be a random sample from

$$f(x|\theta) = \theta x^{\theta-1}, \quad 0 < x < 1, 0 < \theta.$$

Assume that the prior distribution of θ is given by

$$\pi(\theta) = \frac{1}{2}e^{-\theta/2}, \quad 0 < \theta.$$

(a) Find the Bases estimator of θ using the square loss function $l(\theta, a) = (\theta - a)^2$.

(b) Find the Bases estimator of θ using the weighted square loss function $l(\theta, a) = \theta^2(\theta - a)^2$.

12. Let $X_1, X_2, \cdots, X_n$ be a random sample from $N(\mu, 1)$. We would like to test of $H_0 : \mu = 0$ vs. $H_1 : \mu = 2$.

   (a) Derive the most powerful test of level of significance $\alpha = 0.05$.

   (b) Compute the required sample size n so that the test above has type II error probability of 0.1 when $H_1 : \mu = 2$ is true.