Statistics Masters Comprehensive Exam

November 19, 2005

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. (a) Let $X_1, \ldots, X_{36}$ be a random sample from a normal population with mean $\mu = 8$ and variance $\sigma^2 = 25$. Let $\bar{X} = \frac{X_1 + \ldots + X_{36}}{36}$ and let $S^2 = \sum_{i=1}^{36}(X_i - \bar{X})^2/35$. Find $E(\bar{X}S^3)$.

   (b) $X_1, \ldots, X_{100}$ is a random sample from an Exponential population with mean 0.5. Find a such that $P(-a \leq \bar{X} - .5 \leq a) = 0.975$.

2. Let $X_1, \ldots, X_n$ be a random sample from with density

$$f(x; \mu, \sigma) = \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}}, \quad x \geq \mu.$$

Find the MLE's of μ and σ.

3. Let $X_1, \ldots, X_n$ be a random sample from with density

$$f(x;\theta) = \frac{3x^2}{\theta^3}, \quad 0 \leq x \leq \theta.$$

(a) Find a Method of Moments estimator of θ^2.

(b) If $\theta \sim \mathcal{B}eta(2,1)$, find the posterior distribution of θ.

4. Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ population. Derive a likelihood ratio test for testing $H_0 : \sigma^2 \leq \sigma_0^2$ against $H_1 : \sigma^2 > \sigma_0^2$.

5. Suppose $X_1, X_2, X_3, \cdots, X_{72}$ be a random sample with a distribution whose p.d.f. is $f(x) = 2(1-x), 0 < x < 1$.

(a) Find the (approximate) $P(\sum_{i=1}^{72} X_i < 28)$.

(b) Let $W = \min X_i$, a random variable representing the *minimum* value of $X_1, X_2, ..., X_{72}$. Find $P(W < 0.05)$.

6. Let $X_1, X_2, X_3, \cdots, X_n$ be a random sample taken from the distribution with the p.d.f.

$$f(x;\theta) = \theta\ x^{\theta-1}, \quad 0 < x < 1, 0 < \theta < \infty.$$

(a) Find the maximum likelihood estimator of θ.

(b) Find its asymptotic distribution of the MLE.

7. Suppose X_1, X_2 i. i. d. random variables with p.d.f. $f(x) = e^{-x}, x > 0$.

(a) Find the joint p.d.f. of $Y = X_1 + X_2$ and $Z = X_1 - X_2$.

(b) Find the marginal p.d.f. of Z.

8. To test $H_0 : p = 0.5$ against $H_1 : p > 0.5$, we take a random sample of Bernoulli trials $X_1, X_2, X_3, \cdots, X_n$ and use for our test statistic $Y = \sum_{i=1}^{n} X_i$. Let the critical region be defined by $C = \{y : y \geq c\}$.

   (a) If $n = 36$ and $c = 23$, find the type I error probability.

   (b) If $n = 36$ and $c = 23$, find the type II error probability when $p = 0.8$.

   (c) Find the value c so that the type I error probability is about 0.01. [$z_{0.05} = 1.645$, $z_{0.025} = 1.960$, $z_{0.01} = 2.326$, $z_{0.005} = 2.576$]

9. Let $\{X_1, \dots, X_4\}$ be independently and identically distributed normal variables with mean μ and variance σ^2. Put $Y_1 = X_1 + X_2 + X_3$ and $Y_2 = X_2 + X_3 + X_4$.

   (a) Obtain the joint probability density function (pdf) of (Y_1, Y_2).

   (b) What is the pdf of $Z = (Y_1 - Y_2)^2$?

10. Let $\{X_1, \ldots, X_m\}$ be a random sample from the normal distribution with mean μ_1 and variance σ_1^2 and $\{Y_1, \ldots, Y_n\}$ a random sample from the normal distribution with mean μ_2 and variance σ_2^2. Assume that the X_i's are independently distributed of the Y_j's. Put: $\bar{X} = \frac{1}{m}\sum_{i=1}^m X_i, \hat{\sigma}_1^2 = \frac{1}{m-1}\sum_{i=1}^m (X_i - \bar{X})^2$, and $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i, \hat{\sigma}_2^2 = \frac{1}{n-1}\sum_{i=1}^n (Y_i - \bar{Y})^2$.

(a) Define the random variable U by:

$$U = \{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)\}/\sqrt{\frac{\hat{\sigma}_1^2}{m} + \frac{\hat{\sigma}_2^2}{n}}.$$

Prove that if $\frac{\hat{\sigma}_1^2}{m} + \frac{\hat{\sigma}_2^2}{n}$ is distributed as $a\chi_b^2$, where $(a > 0, b > 0)$ are constants, and where χ_b^2 is a central chi-square random variable with degrees of freedom b, then U is distributed as t_b where t_b is a central t random variable with degrees of freedom b.

(b) What are the values of a and b? (Note that a and b are functions of $(\sigma_i^2, i = 1, 2)$.

11. Let $\{X_1, \ldots, X_n\}$ be a random sample from the Poisson distribution with mean θ ($\theta > 0$).

   (a) Obtain a sufficient and complete statistic for θ.

   (b) Derive the UMVUE (Uniformly Minimum Varianced and Unbiased Estimator) of $\phi = e^{-\theta}$.

12. Let $\{X_1, \ldots, X_{16}\}$ be a random sample from the normal distribution with mean θ and variance 4. Consider the null hypothesis $H_0 : \theta = 0$ versus the alternative hypothesis $H_1 : \theta = 2$.

   (a) Derive the size-0.025 MP (Most Powerful) test for testing H_0 vs H_1.

   (b) Assume that the observed sample mean is 2. Based on this observed data, obtain the p-value of your test. From this analysis, what conclusion you will make on H_0?

   (c) Derive the power of your test.