Statistics Masters Comprehensive Exam

March 19, 2005

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Bowl C contains six red chips and four blue chips. Two of these 10 chips are selected at random and without replacement and put in bowl D, which was originally empty. One chip is then drawn at random from bowl D.

   (a) Let A_i be the event that i blue chips were transferred from bowl C to bowl D. Find $P(A_i)$ for $i = 0, 1, 2$.

   (b) Given that this chip drawn from bowl D is blue, find the conditional probability that one red chip and one blue chip were transferred from bowl C to bowl D.

2. In a state lottery game, the player picks 5 integers between 1 and 50, inclusive. The state selects 7 integers between 1 and 50, inclusive. Let X equal the number of integers selected by player that match integers selected by the state.

   (a) Find $P(X = x)$ for $x = 0, 1, 2, 3, 4, 5$.

   (b) If $X = 5, 4$, the player wins \$10,000 and \$1,000, respectively. Compute the expected value of winning for each ticket purchased.

   (c) In addition to above, if $X = 0$, the player receives a free ticket (for the next lottery). Compute the expected value of winning for each ticket purchased.

3. Suppose that $X_1, X_2, \cdots, X_n$ form a random sample from a $B(1, \theta)$ distribution for which the θ is unknown $(0 < \theta < 1)$. Let $T = \sum_{i=1}^{n} X_i$, $n > 3$.

   (a) Find an unbiased estimator of $\theta^2(1-\theta)$.

   (b) Compute the Cramer-Rao lower bound of unbiased estimators of $\theta^2(1-\theta)$.

   (c) Find the U.M.V.U. estimator of $\theta^2(1-\theta)$.

4. Let $\{X_1, \ldots, X_n\}$ be a random sample from the normal density with mean μ and variance σ^2. Derive the size α likelihood ratio test for testing $H_0 : \mu = 0$ vs $H_1 : \mu \neq 0$ when σ^2 is unknown.

5. Suppose that an average of 30 customers per hour arrive at a shop in according with a Poisson process. Let X denote the waiting time in minutes until the third customer arrival. Let Y be the number of customers arrival in 10 minutes. Let W denote the waiting time in minutes for the first customer arrival.

   (a) Find $P(2 \leq Y \leq 6)$?

   (b) Find $P(W \leq 5)$.

   (c) Find $P(5 \leq X \leq 10)$.

6. Let X be a continuous variable with distribution function F.

   (a) Prove that $F(X)$ has a uniform distribution on $(0, 1)$.

   (b) For $0 < \lambda < 1$, find the distribution of $\min\{\frac{F(X)}{\lambda}, \frac{1-F(X)}{1-\lambda}\}$.

7. Let $X_1, \ldots, X_n$ be a independent random variables such that $X_i \sim Normal(\theta a_i, 1)$, where $a_1, \ldots, a_n$ are known constants. Suppose θ has a $N(0, 1)$ prior distribution.

   (a) Find the posterior distribution of θ.

   (b) Using squared error loss, find the Bayes estimator of θ.

8. Let $X_1, \ldots, X_n$ be a random sample an Exponential population with parameter θ. That is,

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0$$

Suppose that the values of $x_1, \ldots x_{n-1}$ of $X_1, \ldots, X_{n-1}$ are observed , but it is known only that $X_n > t$. Using these data,

(a) Find the MLE of θ.

(b) Find the MLE of the median of the distribution,

9. Let $\{X_1, \ldots, X_n\}$ be independently and identically distributed normal variables with mean 0 and variance 1. Put $Y_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $Y_2 = \sum_{i=1}^{n}(X_i - Y_1)^2$.

   (a) Show that Y_1 and Y_2 are independently distributed of each other stochastically.

   (b) What is the sampling distribution of $\frac{n(n-1)Y_1^2}{Y_2}$?

10. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x;\theta) = e^{-(x-\theta)/\sigma}, x > \theta, \sigma > 0$.

   (a) Derive the MLE (Maximum Likelihood Estimator) $\hat{\theta}$ and $\hat{\sigma}$ of θ and σ respectively.

   (b) Show that the estimators from (a) form a set of sufficient and complete statistics for $\{\theta, \sigma\}$.

11. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x, \theta) = \theta^{-1}e^{-x/\theta}, 0 < x; f(x, \theta) = 0$, if $x \leq 0.(\theta > 0)$

   (a) Derive the size-α UMP (Uniformly Most Powerful) test for testing $H_0 : \theta = 1$ vs $H_1 : \theta < 1$.

   (b) Assume that the observed sample mean is 0.98 and n=20. Based on this observed data, obtain the p-value of your test.

12. Let $\{X_1, \ldots, X_{10}\}$ be a random sample from the normal density with mean μ and variance 1. Let the prior distribution of μ be given by $P(\mu) \propto e^{-\frac{1}{8}(\mu-2)^2}$, μ real.

   (a) Assume that the observed value of the sample mean is 1.5. Derive the .95 HPD (Highest Posterior Density) interval of μ.

   (b) Explain how the above HPD interval compares with the 0.95 confidence interval of μ.