Statistics Masters Comprehensive Exam

April 15, 2006

Student Name: ____

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Let X and Y be independent uniform(0,1) random variables.

   (a) Compute $P(XY \leq w)$ and find the pdf of $W = XY$.

   (b) Find the joint pdf of $W = XY$ and $V = Y$ and then find the marginal pdf of $W = XY$.

2. The following questions are related to an experiment that n balls are distributed randomly in to 4 cells.

   (a) Let $n = 6$, find the probability that at least one ball in each cell.

   (b) Let $n = 5$ and X_i be the number of cells containing exactly i balls, find the probability distribution of X_2.

3. Let X be a random random variable from a binomial (n, θ) distribution

$$f(x|\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}, \quad x = 0, 1, 2, \cdots, n$$

and let θ follow a uniform prior pdf

$$\pi(\theta) = 1, \quad 0 < \theta < 1.$$

(a) Find the posterior distribution of θ.

(b) Find the Bayes estimator of θ under the following loss functions:

i. $L(\theta, a) = (\theta - a)^2$.

ii. $L(\theta, a) = \frac{(\theta-a)^2}{\theta(1-\theta)}$.

4. Let X_i, $i = 1, 2, \cdots, n$ be a random sample from from the pdf

$$f(x; \theta) = 2\theta^2 x^{-3}, \quad 0 < \theta < x < \infty.$$

(a) Show that $T = \min_{1 \leq i \leq n} X_i$ is a complete sufficient statistics for θ.

(b) Find the method of moments estimator of θ.

(c) Find the uniformly minimum variance unbiased estimator of θ.

5. Let X_i, $i = 1, 2, \cdots, n$ be iid random variables with $N(\mu, \sigma^2)$ distribution, where both μ and σ^2 are unknown. Derive the likelihood ratio test for $H_0 : \mu = 1$ vs. $H_1 : \mu \neq 1$.

6. Let X be a continuous variable with distribution function F and let λ be a known parameter, such that $0 < \lambda < 1$.

   (a) Prove that $\min\{\frac{F(X)}{\lambda}, \frac{1-F(X)}{1-\lambda}\}$ has a uniform distribution on $(0, 1)$.

   (b) If $U \sim \text{Uniform}(0, 1)$, and is independent of X, find $E(\min[\frac{F(X)}{U}, \frac{1-F(X)}{1-U}])$

7. Let $X_1, \ldots, X_n$ be independent random variables such that $X_i \sim Normal(\theta, \sigma^2/a_i)$, where a_i's are known constants.

   (a) Find the maximum likelihood estimator of θ and σ^2.

   (b) Are these estimates biased or unbiased? Fully justify your answer.

8. Let $(X_1, \ldots, X_n)$ be a random sample from a population with parameter with density

$$f(x|\theta) = e^{-(x-\theta)}, \quad x > \theta$$

We wish to test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. Consider a test which rejects H_0 when $X_{(1)} > C$, where $X_{(1)} = \min(X_1, \ldots, X_n)$.

(a) Find the value of C so that probability of type I error of this procedure is .05.

(b) Find a uniformly most powerful test of these hypotheses at level of significance α.

9. Let $\{X_1, \ldots, X_n\}$ be independently and identically distributed normal variables with mean 0 and variance 1. Put $Y_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $Y_2 = \sum_{i=1}^{n}(X_i - Y_1)^2$.

   (a) Show that Y_1 and Y_2 are independently distributed of each other stochastically.

   (b) What is the sampling distribution of Y_2?

10. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x;\theta) = e^{-\theta}\theta^x/x!, x = 0, 1, \ldots, \infty, \theta > 0$.

(a) Obtain a sufficient and complete statistic for θ.

(b) Derive the UMVUE (Uniformly Minimum Varianced and Unbiased estimator) of $P\{X = 1\}$.

11. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x;\theta) = \theta^{-1}e^{-x/\theta}, 0 < x$; $f(x;\theta) = 0$, if $x \leq 0$. $(\theta > 0)$

   (a) Derive the size-α UMP (Uniformly Most Powerful) test for testing $H_0 : \theta = 1$ vs $H_1 : \theta > 1$.

   (b) Assume that the observed sample mean is 0.98 and $n = 20$. Based on this observed data, obtain the p-value of your test. From on this analysis, what conclusions you will make?

12. Let $\{X_1, \ldots, X_{10}\}$ be a random sample from the normal density with mean μ and variance 1. Let the prior distribution of μ be given by $P(\mu) \propto e^{-\frac{1}{8}(\mu-2)^2}$, μ real.

Assume that the observed value of the sample mean is 1.5. Derive the .95 HPD (Highest Posterior Density) interval of μ.