Statistics Masters Comprehensive Exam

November 17, 2007

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.
3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)
4. You can use the N(0,1) distribution table as attached.

1. Let the joint p.m.f. of X and Y be defined by

$$f(x, y) = \frac{x + y}{32}, \quad x = 1, 2, \quad y = 1, 2, 3, 4.$$

   (a) Find the conditional distribution of Y given $X = x$.

   (b) Find $E(Y|X = 1)$ and $Var(Y|X = 1)$.

   (c) Compute the correlation coefficient between X and Y.

2. A company provides earthquake insurance. The premium X is modeled by the p.d.f. $f(x) = \frac{x}{5^2} e^{-x/5}, 0 < x < \infty$ while the claim Y have the p.d.f. $g(y) = \frac{1}{5} e^{-y/5}, 0 < y < \infty$. If X and Y are independent, find

   (a) the p.d.f. of $Z = X/Y$.

   (b) $P(X > Y)$.

3. Let $X_1, X_2, \ldots, X_n$ be iid Poisson random variables with unknown mean λ. Let $\theta = P(X_1 = 2)$.

   (a) Find a uniformly minimum variance unbiased estimator T_n of θ.

   (b) Find the asymptotic distribution of T_n.

4. If $X_1 \sim B(n_1, p_1)$, $X_2 \sim B(n_2, p_2)$ (binomial distribution) and X_1 is independent of X_2. Suppose x_1 and x_2 are observed and 95% confidence intervals for p_1 and p_2 are $[0.1146, 0.2054]$ and $[0.1824, 0.2776]$, respectively.

   (a) Determine n_1 and n_2 based on the information given above.

   (b) Construct a 95% confidence interval for $p_1 - p_2$.

   (c) Test $H_0 : p_1 = p_2$ vs. $H_1 : p_1 < p_2$ at level $\alpha = 0.05$.

5. Let $\{X_1, \ldots, X_n\}$ be independently and identically distributed normal variables with mean 0 and variance 1. Put $Y_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $Y_2 = \sum_{i=1}^{n}(X_i - Y_1)^2$.

   (a) Show that Y_1 and Y_2 are independently distributed of each other.

   (b) What is the sampling distribution of Y_2?

6. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x; \theta, \lambda) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}(x-\theta)}$, if $\theta < x$ and $= 0$ for $x \geq \theta$.

   (a) Derive the MLE (Maximum Likelihood Estimator) $\hat{\theta}$ and $\hat{\lambda}$ of θ and λ respectively.

   (b) Show that the estimators derived above form a set of sufficient and complete statistics for $\{\theta, \lambda\}$.

7. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x, \theta) = \theta x^{\theta-1}, 0 < x < 1; f(x, \theta) = 0$, for otherwise.

   (a) Derive the size α UMP (Uniformly Most Powerful) test for testing $H_0 : \theta = 1$ vs $H_1 : \theta > 1$.

   (b) Derive the power function of the test derived above.

   (c) Based on observed data $\{x_1, \ldots, x_n\}$, obtain the p-value of the test derived above.

8. Let $\{X_1, \ldots, X_n\}$ be a random sample from the normal density with mean $\mu_1 + \mu_2$ and variance $4\sigma^2$. Let $\{Y_1, \ldots, Y_m\}$ be a random sample from the normal density with mean $\mu_1 - \mu_2$ and variance $9\sigma^2$. Assume that the X_i's are independently distributed of the Y_j's.

   (a) Derive the MLE (Maximum Likelihood Estimator) of $\Theta = \{\mu_1, \mu_2, \sigma^2\}$.

   (b) Assume that a prior distribution of Θ is: $P\{\Theta\} \propto (\sigma^2)^{-1}$. Derive the Bayesian estimator of Θ under the squared loss function.

9. Let X and Y are random variables such that Y and $\frac{X}{Y}$ are independent. Let k be any positive integer.

   (a) Prove that $E\left(\frac{X}{Y}\right)^k = \frac{EX^k}{EY^k}$.

   (b) Prove that $E\left(\frac{Y}{X}\right)^k = \frac{E(X^{-k})}{E(Y^{-k})}$.

10. Let $X_1, \ldots, X_n$ be a independent random variables such that $X_i \sim Normal(\theta a_i, \sigma^2)$, where $\sum_{i=1}^{n} a_i = 1$.

(a) Find the maximum likelihood estimators of θ and σ^2.

(b) When n=2, use the MLE of σ^2 to construct an unbiased estimator of σ^2.

11. Let $X_1, \ldots, X_n$ be a random sample an exponential population with parameter $\theta, n > 2$. That is each X has density

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0$$

Suppose we put a Gamma (α, β) prior on θ.

(a) Show that this prior is conjugate.

(b) Find the Bayes estimator of θ if we use the loss function $L(\theta, a) = (\theta - a)^2/\theta$.

12. Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x;\theta) = \theta x e^{-\frac{\theta x^2}{2}}, \qquad x > 0.$$

(a) Find a sufficient statistic for θ.

(b) Construct a uniformly most powerful level α test for testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

(c) If $n = 400$, find approximately the power of the test at $\theta = 2$, when $\alpha = 0.05$

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998