Statistics Masters Comprehensive Exam

April 7, 2007

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Let U_1 and U_2 be independent uniform(0,1) random variables. Let $R = \sqrt{-2\ln(U_1)}$ and $\theta = 2\pi U_2$. Consider the transformation $X = R\cos\theta$ and $Y = R\sin\theta$.

   (a) Find the p.d.f. of θ.

   (b) Find the p.d.f. of R.

   (c) Show that X and Y are two independent N(0,1) random variables.

2. A box contains 3 dimes and 2 quarters. Two coins are selected randomly without replacement. The amount selected, S, is recorded and the coins are returned into the box. This experiment is performed 100 times. Let X be the total amount recorded.

   (a) Find the probability distribution of S.

   (b) Find the mean and the variance of S.

   (c) Find $P(X \leq 30 \text{ dollars})$.

3. Let $X_1, X_2, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$.

   (a) Assume that σ^2 is known to be 1. Derive the most powerful test of $H_0 : \mu = 0$ vs. $H_1 : \mu = 1$ of level of significance $\alpha = 0.05$.

   (b) If σ^2 is unknown, how would you test $H_0 : \sigma^2 \leq 1$ vs. $H_1 : \sigma^2 > 1$? Justify your procedure.

4. Let X_1, X_2 denote two independent random variables, each with $\chi^2(2)$, a Chi-square distribution with 2 degrees of freedom.

   (a) Find the joint p.d.f. of $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$.

   (b) Find the marginal p.d.f. of each of Y_1 and Y_2.

   (c) Are Y_1 and Y_2 independent? Explain.

5. Let X, Y have a bivariate normal distribution with $E(X) = 70$, $\text{Var}(X) = 100$, $E(Y) = 80$, $\text{Var}(Y) = 169$, and correlation coefficient $\rho = 5/13$.

(a) Find $E(Y|X = 72)$ and $\text{Var}(Y|X = 72)$

(b) Find $P(Y \leq 84|X = 72)$.

(c) Find $P(X > Y)$.

6. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x;\theta) = \frac{1}{\theta}x^{(1-\theta)/\theta}$, where $0 < x < 1$ and $0 < \theta < 1$.

   (a) Find the maximum likelihood estimator of θ, $\hat{\theta}$.

   (b) Is $\hat{\theta}$ an unbiased estimator for θ ? Prove your claim.

7. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x, \theta) = \frac{1}{\theta}$ if $0 < x \leq \theta, \theta > 0$ and $= 0$ for otherwise.

   (a) Obtain the MLE (Maximum Likelihood Estimator) $\hat{\theta}_M$ of θ.

   (b) Show that $\hat{\theta}_M$ is a sufficient and complete statistic.

   (c) Derive the UMVUE (Uniformly Minmum Variances Unbiased estimator) $\hat{\theta}$ of θ.

   (d) Compare $\hat{\theta}_M$ with $\hat{\theta}$.

8. Let $\{X_1, \ldots, X_m\}$ be a random sample from the normal distribution with mean μ_1 and variance σ^2 and $\{Y_1, \ldots, Y_n\}$ a random sample from the normal distribution with mean μ_2 and variance σ^2. Assume that the X_i's are independently distributed of the Y_j's.

   (a) Derive a $(1-\alpha)\%$ Confidence Interval for $\delta = \mu_1 - \mu_2$ in terms of central t-distribution. (σ^2 is unknown.)

   (b) Derive a $(1-\alpha)\%$ Confidence Interval for σ^2 in terms of central chi-square distribution. (You have to use both samples.)

9. Let $\{X_1, \ldots, X_n\}$ be a random sample from the distribution with pdf $f(x, \theta) = \frac{1}{\theta} e^{-x/\theta}, x > 0, \theta > 0$.

   (a) Derive the size-0.05 UMP (Uniformly Most Powerful) test for testing $H_0 : \theta = 1$ vs $H_1 : \theta > 1$.

   (b) Derive the size-0.05 UMP (Uniformly Most Powerful) test for testing $H_0 : \theta = 1$ vs $H_2 : \theta < 1$.

   (c) Does the size-0.05 UMP (Uniformly Most Powerful) test for testing $H_0 : \theta = 1$ vs $H_3 : \theta \neq 1$ exit ? Why ?

10. Let $X_1, \dots, X_n$ be a random sample from a population with density

$$f(x|\mu, \sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, \quad x \geq \mu.$$

(a) Find the Method of Moments estimators (MME) of μ and σ.

(b) Are any of these estimators functions of the jointly sufficiently statistics for (μ, σ)?

(c) Compare the MME with the maximum likelihood estimators of (μ, σ).

11. Consider the simple linear models

$$Y_{ij} = \beta_i x_{ij} + \varepsilon_{ij},$$

$i = 1, 2;\ j = 1, \ldots, n$. Assume that ε_{ij}'s are independent and distributed as $N(0, \sigma^2)$ for all i, j.

(a) Find the MLE's of β_1, β_2 and σ^2.

(b) Construct a likelihood ratio test of $H_0 : \beta_1 = \beta_2$ versus $H_1 : \beta_1 \neq \beta_2$.

12. Let X_1 and X_2 be a random sample from Normal$(0, 1/\theta)$, and assume that θ has an Exponential(1) prior distribution.

   (a) Find the posterior distribution of θ.

   (b) Using squared error loss, find the Bayes estimator of θ.

   (c) Find c such $P(\theta > c | X_1^2 + X_2^2 = 1) = 0.05$.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998