Statistics Masters Comprehensive Exam

November 15, 2008

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Let X be distributed as a normal random variable with mean 0 and variance σ^2. Let$Y$ be a random variable such that $\frac{Y}{\sigma^2}$ is distributed as central chi-square with n degrees of freedom independently of X.

   (a) Derive the probability density function (pdf) of $Z = \frac{X^2}{Y}$.

   (b) What is the pdf of $\sqrt{Z}$?

2. Let $\{X_1, \ldots, X_n\}$ be a random sample from the population with probability density $f(x;\theta,\sigma) = exp\{-\frac{1}{\sigma}(x-\theta)\}, \theta < x, \theta > 0, \sigma > 0$.

   (a) Show that the jointly sufficient and complete statistics for (θ, σ) are given by $(X_{(1)}, S)$ where $\{X_{(1)} = \min(X_1, X_2, \ldots, X_n)$ and $S = \sum_{j=1}^{n}(X_j - X_{(1)})\}$.

   (b) Derive the UMVUE (Uniformly Minimum Varianced Unbiased estimator) of θ. What is the UMVUE of σ?

3. Let $\{X_1, \ldots, X_n\}$ be a random sample from the population with probability density $f(x, \theta) = \theta\ x^{\theta-1}, 0 < x < 1, \theta > 0$. Let the prior distribution of θ be given by $P(\theta) \propto \theta^2 e^{-2\theta}, \theta > 0$.

   (a) Obtain the posterior distribution of θ.

   (b) Derive the Bayesian estimator of θ under the squared loss function.

4. Let $\{X_1, \ldots, X_n\}$ be a random sample from the population with probability density $f(x, \theta) = \theta^{-1} e^{-x/\theta}, 0 < x, \theta > 0$.

   (a) Obtain the level-α UMP (Uniformly Most Powerful) test for testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

   (b) Show that the level-α UMP (Uniformly Most Powerful) test for testing $H_0 : \theta = 1$ versus $H_1^{(*)} : \theta \neq 1$ does not exist.

5. Let X_1 and X_2 be independent random variables with $X_i \sim \mathcal{P}\text{oisson}(\lambda_i), i = 1, 2$. If $Y = X_1 + X_2$, and $Z = X_1 X_2$

   (a) Find $P(X_1 = n | Y = K)$

   (b) Find $Cov(Y, Z)$

6. Let $X_1, \dots, X_n$ be a random sample from a population with density

$$f(x;\theta) = 3\theta x^2 e^{-\theta x^3}, \qquad x > 0.$$

(a) Derive the UMP size α test for testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$. Give the critical values of your test in terms of standard tabulated percentiles.

(b) Determine a $100(1-\alpha)\%$ confidence interval for θ

7. Suppose that X has the following distribution:

x	0	2	4	6
P(X=x)	0.1	0.4	0.2	0.3

You take a random sample of size 400 from the population. Let S denote the sample sum. Find b such that $P(S > b) = 0.975$.

8. Consider the simple linear model

$$Y_i = \alpha + \beta x_i + \epsilon_i,$$

where x_i's are fixed covariates, ϵ_i's are independent normal random variables with mean 0 and unknown variance σ^2, $i = 1, \ldots, n$.

(a) Find the maximum likelihood estimator of α, β.

(b) Are the MLE's of α and β sufficient statistics for α and β ?

(c) Are these MLE's unbiased estimator of α and β?

9. Let $X_1, X_2, \cdots, X_n$ be a random sample taken from the distribution with the p.d.f.

$$f(x;\theta) = 1/\theta \; x^{(1-\theta)/\theta}, \quad 0 < x < 1, 0 < \theta < \infty.$$

(a) Find the moment estimator of θ. Is it unbiased ?

(b) Find the maximum likelihood estimator of θ. Is it unbiased ?

(c) Find Rao-Cramér lower bound of any unbiased estimator $\hat{\theta}$ for θ.

10. Let X have a beta distribution with parameters α and β

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1, \alpha > 0, \beta > 0,$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$.

(a) Show that $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$, for $\alpha > 1$.

(b) Show the mean and variance of X are

$$\mu = \frac{\alpha}{\alpha+\beta}, \qquad \text{and } \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}.$$

11. Let $f(x,y) = 3/2, x^2 \leq y \leq 1, 0 \leq x \leq 1$ be the joint p.d.f. of X and Y. Find

(a) $P(1/2 \leq Y)$.

(b) $P(X \geq Y)$.

(c) Marginal p.d.f. of X and Y.

12. Let $X_1, X_2, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$. We would like to test of $H_0 : \mu = 0$ vs. $H_1 : \mu = 1$.

    (a) If $\sigma^2 = 1$ is known, derive the most powerful test of level of significance $\alpha = 0.05$.

    (b) If σ^2 is unknown, show how to find a reasonable test of level of significance $\alpha = 0.05$.

Table of $P(Z < z)$, $Z \sim N(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998