Statistics Masters Comprehensive Exam

April 12, 2008

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Let $f(x, y) = c$, $x^2 \leq y \leq 1$, $0 \leq x \leq 1$, be the joint p.d.f. of X and Y. Here c is a constant to be determined. Find

   (a) The constant c so that $f(x, y)$ is a joint p.d.f.

   (b) $P(0 \leq X \leq 1/2)$

   (c) $P(0 \leq X + Y \leq 1)$

2. A box contains three coins with a head on each side, four coins with a tail on each side, and three fair coins.

   (a) If one of these ten coins is selected at random and tossed once, what is the probability that a head will be obtained?

   (b) Same experiment as above and a head is observed. If the same coin were tossed again, what would be the probability of obtaining another head?

3. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x;\theta) = \frac{1}{2}exp\{-(x-\theta)/2\}, x > \theta$.

   (a) Show that a sufficient and complete statistic for θ is $X_{(1)} = \min(X_1, \ldots, X_n)$.

   (b) Derive the UMVUE (Uniformly Minimum Varianced Unbiased Estimator) of θ.

4. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x, \theta) = \theta \ x^{\theta-1}, 0 < x < 1, \theta > 0$.

(a) Derive the size α UMP (Uniformly Most Powerful) test for testing $H_0 : \theta = 1$ vs $H_1 : \theta > 1$.

(b) Derive the sampling distribution of your testing statistic under H_0. [Hint: what is the distribution of $Y_i = -\ln(X_i)$?]

(c) Based on observed data $\{0.413, 0.628, 0.747, 0.832, 0.951\}$, explain how to obtain the p-value of your test.

5. Let $\{X_1, \ldots, X_n\}$ be a random sample from the Bernoulli density $f(x) = \theta^x(1-\theta)^{1-x}, x = 0, 1; 0 < \theta < 1$.

(a) Derive the UMVUE $\hat{\theta}_M$ of θ.

(b) Assuming a prior distribution of θ as $P(\theta) = 1, 0 < \theta < 1$, derive the Bayes estimator $\hat{\theta}_B$ of θ under squared loss function.

(c) Assume that n=10 and that the true value of θ is a number between 0.3 and 0.7. Based on the Mean-Squared error criteria, which estimator of $\{\hat{\theta}_M, \hat{\theta}_B\}$ of θ would you prefer? Why?

6. Let X, Y be two independent random variables with a uniform $U(-1, 3)$ distribution.

(a) Find the p.d.f. of $Z = X^2$.

(b) Find the p.d.f. of $D = X - Y$.

7. If $X_1 \sim B(n_1, p_1)$, $X_2 \sim B(n_2, p_2)$ (binomial distribution) and X_1 is independent of X_2. Suppose x_1 and x_2 are observed and 95% confidence intervals for p_1 and p_2 are $[0.1446, 0.2554]$ and $[0.1679, 0.2765]$, respectively.

   (a) Determine n_1 and n_2 based on the information given above.

   (b) Construct a 95% confidence interval for $p_1 - p_2$.

   (c) Test $H_0 : p_1 = p_2$ vs. $H_1 : p_1 < p_2$ at level $\alpha = 0.05$.

8. Let random variable Y have a Weibull density function

$$f(y) = \frac{by^{b-1}e^{-(y/\theta)^b}}{\theta^b}, y > 0.$$

(a) Let $X = Y^b$. Prove that X has an exponential distribution.

(b) Suppose b is a known fixed constant. Construct a UMP test for testing

$$H_0 : \theta = 1 \text{ vs } H_1 : \theta > 1.$$

Give your critical value in terms of a chi-square percentile.

9. Let $X_1, \ldots, X_n$ be a random sample from a population with a Uniform$(\theta - 2, \theta + 2)$ distribution

   (a) Find sufficient statistics for θ.

   (b) Are these sufficient statistics complete? Provide a proof to justify your answer.

   (c) Find a maximum likelihood estimator of θ.

   (d) Is the MLE unique?

10. Let $X_1, \ldots X_n$ be a random sample from a population with Bernoulli (θ) population. Let r and s be non-negative integers with $0 \leq r + s \leq n$.

   (a) Find an unbiased estimator of $\theta^r(1-\theta)^s$.

   (b) Find the Cramer-Rao lower bound for the variance of any unbiased estimator of $\theta^r(1-\theta)^s$.

   (c) Find the UMVUE of $\theta^r(1-\theta)^s$.

11. Consider the simple linear models

$$Y_{ij} = \beta_i x_{ij} + \varepsilon_{ij},$$

$i = 1, 2$; $j = 1, \ldots, n$. Assume that ε_{ij}'s are independent and distributed as $N(0, \sigma^2)$ for all i, j.

(a) Find the MLE's of β_1, β_2 and σ^2.

(b) Construct a likelihood ratio test of $H_0 : \beta_1 = \beta_2$ versus $H_1 : \beta_1 \neq \beta_2$.

12. Let X_1 and X_2 be a random sample from Normal$(0, 1/\theta)$, and assume that θ has an Exponential(1) prior distribution.

    (a) Find the posterior distribution of θ.

    (b) Using squared error loss, find the Bayes estimator of θ.

    (c) Find c such $P(\theta > c | X_1^2 + X_2^2 = 1) = 0.05$.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998