Statistics Masters Comprehensive Exam

April 18, 2009

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.
3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)
4. You can use the N(0,1) distribution table as attached.

1. Let $\{X_1, X_2, X_3\}$ be independently and identically distributed normal variables with mean μ and variance σ^2. Put $Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$.

   (a) Obtain the joint probability density function (pdf) of (Y_1, Y_2).

   (b) What is the pdf of $Z = (Y_1 - Y_2)^2$?

2. Let $\{X_1, \ldots, X_n\}$ be a random sample from the point binomial distribution with pdf given by $f(x, \theta) = \theta^x (1-\theta)^{1-x}, x = 0, 1$ and $0 < \theta < 1$. Let the prior distribution of θ be given by $P(\theta) = \frac{1}{B(0.5,0.5)} \theta^{-0.5} (1-\theta)^{-0.5}, 0 < \theta < 1$.

   (a) Derive the posterior distribution of θ.

   (b) Assume a squared loss function $l(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$ for estimator $\hat{\theta}$ of θ and assume $(n = 20, y = \sum_{i+1}^{n} x_i = 10)$, where the x_i's are the observed values of X_i's respectively. Obtain the Bayese estimate of θ.

3. Let $\{X_1, \ldots, X_n\}$ be a random sample from the Poisson distribution with mean θ ($\theta > 0$).

    (a) Obtain a sufficient and complete statistic for θ.

    (b) Derive the UMVUE (Uniformly Minimum Varianced and Unbiased Estimator) of $\phi = e^{-\theta}$.

4. Let $\{X_1, \ldots, X_{16}\}$ be a random sample from the normal distribution with mean θ and variance 4. Consider the null hypothesis $H_0 : \theta = 0$ versus the alternative hypothesis $H_1 : \theta = 2$.

   (a) Derive the size-0.025 MP (Most Powerful) test for testing H_0 vs H_1.

   (b) Assume that the observed sample mean is 2. Based on this observed data, obtain the p-value of your test. From this analysis, what conclusion you will make on H_0?

5. In order to decide the appropriate amount to charge as premium, insurance companies often use the exponential principle defined as follows: If X is the random amount that it will have to pay in claims, then the premium charged by the insurance company should be

$$P = \frac{1}{a} \ln(E[e^{aX}]),$$

where $a > 0$ is a fixed specified constant. Suppose that an insurance company assumes that X has a uniform distribution distribution on $[0, \theta]$.

(a) Find P.

(b) An insurance company wishes to find a maximum likelihood estimator $\hat{P}$ of P, by taking a random sample $X_1, \ldots, X_n$ from a large set of previous payments. Assuming that the X's are a random sample from $Uniform[0, \theta]$, where θ is unknown. Find $\hat{P}$.

6. Let $X_1, \dots, X_n$ be a random sample from a geometric distribution with parameter θ. That is,

$$f(x|\theta) = (1-\theta)^{x-1}\theta, \quad x = 1, 2, 3, \dots$$

(a) Find a minimal sufficient statistic $T(\mathbf{X})$ for θ.

(b) Prove that $T(\mathbf{X})$ is a complete sufficient statistic for θ.

(c) Find a uniformly minimum variance unbiased estimator (UMVUE) for θ.

(d) Is this estimator a unique UMVUE? Justify your answer carefully

7. Let X and Y be jointly distributed random variables. If $X|Y=y \sim Poisson(y)$, and $Y \sim Exponential(\lambda)$

   (a) Find the marginal distribution of X

   (b) Find $P(Y > a|X \leq 1)$

   (c) Let $(X_1, X_2, X_3) \sim \text{Trinomial}(n, p_1, p_2, p_3)$, that is,

   $$P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3},$$

   where $p_2 = 1 - p_1 - p_2,\ X_3 = n - X_1 - X_2$, find $P(X_1 = k|X_1 + X_2 = m)$.

8. Let X, Y be independent random variables with $X \sim Gamma(2, \theta)$ and $Y \sim Gamma(3, \theta)$. Let $V = X + Y$ and $W = Y/(X + Y)$.

   (a) Find the joint density function of (V, W).

   (b) Find the marginal density of W.

9. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with mean θ. We wish to test

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = 2.$$

For the procedure $\delta(\mathbf{X})$: Reject H_0 if $minimum(X_1, \ldots, X_n) \leq 1$.

(a) Find the probability of Type I error.

(b) Find the probability of Type II error

(c) For

$$H_0 : \theta \leq 1 \quad \text{versus} \quad H_1 : \theta > 1$$

find the power function of the procedure δ, and find its size.

(d) Derive the uniformly most powerful test of $H_0 : \theta \leq 1 \quad \text{versus} \quad H_1 : \theta > 1,.$

(e) Compare the UMP test with δ.

10. Let Let $X_1, \ldots, X_n$ be a random sample from a distribution with p.d.f. $f(x;\theta)$ to be specified.

   (a) If $f(x;\theta) = \frac{1}{\theta} x^{\frac{1-\theta}{\theta}}, 0 < x < 1, \theta > 0$. Find $\hat{\theta}$, the maximum likelihood estimator of θ. Compute $E(\hat{\theta})$.

   (b) If $f(x;\theta) = \frac{1}{\theta}, 0 < x < \theta, \theta > 0$. Find $\hat{\theta}$, the maximum likelihood estimator of θ. Compute $E(\hat{\theta})$.

11. Let $f(x, y) = c, x^2 \leq y \leq 2, 0 \leq x \leq 1$ be the joint p.d.f. of X and Y, where c is a constant to be determined. Find

   (a) constant c.

   (b) $P(1/2 \leq Y \leq 1)$.

   (c) $P(X \geq Y)$.

   (d) marginal p.d.f. of X.

12. If $X_1 \sim B(n_1, p_1)$, $X_2 \sim B(n_2, p_2)$ (binomial distribution) and X_1 is independent of X_2. Suppose x_1 and x_2 are observed and 95% confidence intervals for p_1 and p_2 are $[0.1477, 0.2523]$ and $[0.1970, 0.3030]$, respectively.

   (a) Determine n_1, x_1 and n_2, x_2 based on the information given above.

   (b) Construct a 95% confidence interval for $p_1 - p_2$.

   (c) Test $H_0 : p_1 = p_2$ vs. $H_1 : p_1 < p_2$ at level $\alpha = 0.05$.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998