Statistics Masters Comprehensive Exam

April 3, 2010

Student Name: ____

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

1. Suppose that the joint p.d.f. of two random variables X and Y is $f(x, y) = x+y, \quad 0 \leq x, y \leq 1$.

   (a) Find the marginal p.d.f. of X.

   (b) Find $P(2X + Y \leq 1)$.

   (c) Find the p.d.f. of $Z = X + Y$.

2. Let X and Y be two random variables with joint density

$$f(x,y) = \begin{cases} 1, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Define $U = X + Y, V = X/(X+Y)$.

(a) Find the the joint density of U and V.

(b) Find the the marginal density of U.

(c) Find the the marginal density of V.

3. Let $(X_1, X_2, X_3) \sim \text{Trinomial}(n, p_1, p_2, p_3)$, that is,

$$P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3},$$

where $p_3 = 1 - p_1 - p_2,\ x_3 = n - x_1 - x_2$, find $P(X_1 = k | X_3 = m)$.

4. Suppose $X_1, X_2, X_3, ...X_{64}$ be a random sample with a *beta* distribution with the p.d.f. $f(x) = 2x, 0 < x < 1$.

   (a) Let Y be the number of these random variables $(X_i, i = 1, 2, ..., 64)$ whose values less than 0.5. Approximate $P(Y \leq 20)$.

   (b) Approximate $P(\sum_{i=1}^{30} X_i > \sum_{i=31}^{64} X_i)$.

5. Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with pmf

$$f(x) = \theta^x(1-\theta)^{1-x}, x = 0, 1; 0 < \theta < 1.$$

Find UMVUEs for

(a) θ

(b) θ^2

(c) $\theta(1-\theta)$.

6. Let $X_1, \dots, X_n$ be a random sample from a population with density

$$f(x|\theta) = \begin{cases} \left(\frac{\theta}{x}\right)^{\theta+1} & , \quad \text{if} \quad x > \theta \\ 0 & , \quad \text{if otherwise,} \end{cases}$$

where $\theta > 4$.

(a) Find the maximum likelihood estimator (MLE) of θ^4.

(b) Is the above MLE a minimal sufficient statistics? Explain fully.

7. Let X_1, X_2 be independent random variables.

(a) If $X_1 \sim Poisson(5)$ and $X_2 \sim Poisson(2)$, find $P(X_1 = 1 | X_1 + X_2 = 2)$.

(b) If $X|\theta \sim Poisson(\theta)$ and $\theta \sim Gamma(\alpha, \beta)$, find $E(\theta^4 | X = 2)$.

8. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli (θ), and suppose we put a Beta(α, β) prior distribution on θ.

   Find Bayes estimators of θ using loss functions

   (a) $L(\theta, a) = (\theta - a)^2$

   (b) $L(\theta, a) = \frac{(\theta-a)^2}{\theta(1-\theta)}$,

   (c) Show that with the second loss function, the MLE of θ is the Bayes estimator of θ, when $\alpha = \beta = 1$.

9. Let $X_1, \ldots, X_n$ be a random sample from

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & , \quad \text{if} \quad 0 < x < 1, \theta > 0, \\ 0 & , \quad \text{if otherwise,} \end{cases}$$

where $\theta > 1$.

(a) Find MP test of size α for testing $H_0 : \theta = 1$ vs. $H_0 : \theta = 2$.

(b) Find UMP test of size α for testing $H_0 : \theta = 1$ vs. $H_0 : \theta > 1$.

(c) Obtain the power function and sketch its graph.

10. Let $\{X_1, \ldots, X_n\}$ be a random sample from the Bernoulli distribution with density $f(x, \theta) = \theta^x(1-\theta)^{1-x}, x = 0, 1$, where $0 < \theta < 1$. Let the density of the prior distribution be given by $P(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}, 0 \leq \theta \leq 1$, where $a > 0, b > 0$.

   (a) Given the loss function as $l(\theta, s) = (s - \theta)^2$ for the estimator s of θ, derive the Bayese estimator of θ.

   (b) Derive the maximum likelihood estimator (MLE) $\hat{\theta}$ of θ and compare the MLE with the Bayese estimator.

11. Let $X_1, X_2, \cdots, X_n$ be a random sample taken from the distribution with the p.d.f.

$$f(x;\theta) = 1/\theta \; x^{1/\theta - 1}, \quad 0 < x < 1, 0 < \theta < \infty.$$

(a) Find the p.d.f. of $Y = -\ln X$, where X is a random variable with p.d.f. $f(x;\theta)$ given above.

(b) Find the moment estimator of θ.

(c) Find the maximum likelihood estimator of θ.

(d) Find Rao-Cramér lower bound of any unbiased estimator $\hat{\theta}$ for θ.

12. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independently and identically distributed as (X, Y), where (X, Y) follows a bivariate normal distribution with means $EX = \mu_1$ and $EY = \mu_2$ and with the variances $Var(X) = \sigma_1^2$ and $Var(Y) = \sigma_2^2$ respectively. Assume that X is un-correlated with Y.

   (a) Derive the likelihood ratio test (LRT) for testing $H_0 : \sigma_1^2 = \sigma_2^2$ against the alternative hypothesis $H_1 : \sigma_1^2 \neq \sigma_2^2$.

   (b) Derive the probability distribution of your test statistic under H_1. What is the power function of the LRT test?