Statistics Masters Comprehensive Exam

July 31, 2012

Student Name: ____________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.
3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)
4. You can use the N(0,1) distribution table as attached.

1. Let $X_1, \cdots, X_n$ be a random sample from the probability distribution with density

$$f(x; \alpha, \beta) = \frac{1}{\beta} e^{-\frac{1}{\beta}(x-\alpha)}, \quad x \geq \alpha, \beta > 0.$$

   (a) Derive the joint sufficient and complete statistics for (α, β).

   (b) Derive the UMVUE (Uniformly Minimum Variance Unbiased Estimator) of α.

   (c) Find the UMVUE of β.

2. Z_1 and Z_2 are a random sample of size 2 from $N(0,1)$. (normal population with mean $= 0$ and variance $= 1$). X_1 and X_2 are a random sample of size 2 from $N(1,1)$. Z_1, Z_2 are independent of X_1, X_2. Find the distribution of the following.

   (a) $\bar{X} + \bar{Z}$

   (b) $\frac{Z_1+Z_2}{\sqrt{\frac{(X_2-X_1)^2+(Z_2-Z_1)^2}{2}}}$

   (c) $\frac{(X_1-X_2)^2+(Z_1-Z_2)^2+(Z_1+Z_2)^2}{2}$

   (d) $\frac{(X_1+X_2-2)^2}{(X_2-X_1)^2}$

3. $X_1, X_2, \ldots, X_n$ are iid from $\theta e^{-\theta x} \quad x > 0$.

   (a) Find the UMVUE of $\frac{1}{\theta}$. Find the variance of your estimator.

   (b) Find the UMVUE of θ. Find the variance of your estimator.

4. $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the order statistics from independent random variables $X_1, X_2, \ldots, X_n$ with pdf

$$f(x) = \begin{cases} \beta e^{-\beta x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $X_{(r)}$ and $X_{(s)} - X_{(r)}$ are independent for $s > r$.

(b) Find the pdf of $X_{(r+1)} - X_{(r)}$.

5. An urn has M white marbles and $10 - M$ black marbles. To test $H_0 : M = 5$ versus $H_0 : M = 6$, three marbles are drawn from the urn without replacement. The null hypothesis is rejected if 2 or 3 white marbles are drawn. Otherwise H_0 is accepted. Find the size of the test and its power.

6. Let $X_1, \dots X_n$ be a random sample from a population with the Gamma(α, θ), density

$$f(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\theta x), \quad x > 0.$$

(a) Find $E(X^s e^{-rX})$.

(b) Find Method of Moments estimators of α and θ.

7. Let $X_1, \ldots, X_4$ be a random sample an Normal$(0, \sigma^2)$ population. We wish to test $H_0 : \sigma^2 = 1$ versus $H_1 : \sigma^2 = 4$.

   (a) Consider the following tests procedures

      i. Reject H_0 if $\sum_{i=1}^{4} X_i^2 > C_1$.

      ii. Reject H_0 if $\min\{X_1^2, \ldots, X_4^2\} > C_2$.

   (b) Find C_1 and C_2 so that the probability of Type I error is .05

   (c) Calculate the power of the tests and compare the two tests.

8. Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x|\theta) = \begin{cases} 2x\theta^2 & , \quad \text{if} \quad 0 < x < \frac{1}{\theta} \\ 0 & , \quad \text{if otherwise,} \end{cases}$$

(a) Find the maximum likelihood estimator (MLE) of θ^4.

(b) Find a sufficient statistic for θ.

(c) Is the above MLE a minimal sufficient statistics? Explain fully.

9. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli (θ), and suppose we put a Beta(α, β) prior distribution on θ. Find Bayes estimators of θ using loss functions

(a) $L(\theta, a) = (\theta - a)^2$

(b) $L(\theta, a) = \frac{(\theta - a)^2}{\theta^2(1-\theta)}$

(c) Is there any relationship between the MLE of θ and above the Bayes estimator of θ? Explain fully.

10. Let X and Y be independent uniform(0,1) random variables.

(a) Find the cdf of $W = XY$, $P(XY \leq w)$.

(b) Find the pdf of $W = XY$.

11. Let X and Y have joint pdf

$$f(x, y) = C(x + 2y), \quad -1 \leq x \leq 2, \quad 0 < y < 1.$$

(a) Find the constant C.

(b) Find the marginal pdf of X.

(c) Find the pdf of $Z = X^2$.

12. Let $X_1, \ldots, X_n$ is a random sample from Normal(μ_1, σ_1^2) population; and $Y_1, \ldots, Y_m$ is a random sample from Normal(μ_2, σ_2^2) population. Write the best tests of the following hypotheses:

    (a) $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$. (assume $\sigma_1 = \sigma_2 = \sigma$, and σ is unknown.)

    (b) $H_0 : \sigma_1 = \sigma_2$ versus $H_1 : \sigma_1 \neq \sigma_2$.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999