SPRING 2010 COMPREHENSIVE EXAM FOR MASTERS IN MATHEMATICS FOR TEACHERS

A passing grade on this exam can be obtained by answering 5 questions correctly.

Question 1

On the real numbers define a binary operation as a*b = a^b . Answer the following questions:

(1) Is * associative?
(2) Is * commutative?
(3) Does * have an identity?
(4) Let G = $\mathbb{Q}$, the rational numbers. Define a*b= a + b + $\frac{1}{2}$. Show that G, * is an abelian group.
(5) On $\mathbb{Z}$ define a mapping $f : \mathbb{Z} \to \mathbb{Z}$ by f(x) = 3x. Answer the following questions:
   (a) Is f a homomorphism of groups?
   (b) Is f injective?
   (c) Is f surjective?
(6) Let

$$\alpha = \begin{pmatrix} 1 & 2 & 5 & 4 & 3 \end{pmatrix}$$
$$\beta = \begin{pmatrix} 1 & 2 & 5 & 4 & 3 \end{pmatrix}$$

be permutations in S_5. Find the following:
   (a) $\beta \circ \alpha$
   (b) β^{-1}
   (c) Write α as the product of transpositions.

Question 2

1.(a) Show that

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} \right\}$$

is a linearly independent subset of $\mathbb{R}^3$.

1.(b) Find a vector u in $\mathbb{R}^3$ such that

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}, u \right\}$$

is a basis for $\mathbb{R}^3$.

1.(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

$$T \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix},$$

$$Tu = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

Find a basis for the kernel of T.

Question 3

(1) Discuss some reasons for the Greek Miracle.

(2) Contrast Albrecht Durer's Melancholia with M. C. Escher's Relativity.

(3) Write 56 as the sum of three or fewer triangular numbers.

Question 4

(1) Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ if x is rational and $f(x) = -x$ if x is irrational is continuous at $x = 0$ and discontinuous at every $x \neq 0$.

(2) Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable at exactly $x = 0$ and not differentiable for every $x \neq 0$? Explain your answer.

Question 5

(1) Define what it means for a sequence $(a_n)_{n=1}^{\infty}$ to diverge to $+\infty$. We write this $\lim_{n\to\infty} a_n = +\infty$.
(2) Prove that if (a_n) is non-decreasing and there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} a_{n_k} = +\infty$$

then

$$\lim_{n\to\infty} a_n = +\infty.$$

Question 6

Define

$$s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

so that s_n is the n-th partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.
(1) Show that

$$s_9 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} > \frac{9}{10}.$$

(2) Generalize this to show that

$$s_{10^k - 1} > k\frac{9}{10}.$$

(3) Hence show that $\lim_{n\to\infty} s_n = +\infty$.

Question 7

(1) Define what it means for f to be differentiable at a.
(2) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable at a. Show that

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

(3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. What does the Fundamental Theorem of Calculus say about $\int_a^b f'$?
(4) Prove that if $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable then

$$\int_a^b f \cdot g' = f(b) \cdot g(b) - f(a) \cdot g(a) - \int_a^b f' \cdot g$$

You may use the Fundamental Theorem of Calculus without proof.

Question 8

(1) Prove the *Binomial Theorem*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

(2) Use the Binomial Theorem to prove that for any $p \in \mathbb{R}$

$$1 = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

and

$$(1-2p)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k p^k (1-p)^{n-k}$$

(3) Hence prove that

$$\frac{1-(1-2p)^n}{2} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} p^k (1-p)^{n-k}.$$

(4) Consider the Binomial random variable of the number of successes from n independent Bernoulli trials with the probability of success p. Show that if n is even and $p \neq \frac{1}{2}$ then you are more likely to get an even number of successes than an odd number of successes.

Question 9

(1) Define the linear span of vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$.
(2) Define the row space of a matrix A.
(3) Show that the row space of a matrix is invariant under row operations.
(4) Use row reduction to find a basis for the subspace spanned by the vectors
$(1,2,3,4,5), (6,4,3,2,0), (0,2,9,10,18), (1,0,1,0,1), (7,-4,-4,-14,-15)$.
(5) Explain the difference between using a row space and a column space when using row reduction of a matrix to find a basis for the span of a set of vectors.

Question 10

(1) Let a,b,and c be positive integers. Show that the Diophantine equation

$$ax + by = c$$

has solutions $x, y \in \mathbb{Z}$ if and only if $\gcd(a, b)|c$.

(2) Find a solution to the Diophantine equation

$$156x + 91y = 39.$$

(3) Find *all* solutions to the Diophantine equation

$$156x + 91y = 39.$$

Question 11

(http://lib.stat.cmu.edu/DASL/) An educator conducted an experiment to test whether new directed reading activities in the classroom will help elementary school pupils improve some aspects of their reading ability. She arranged for a third grade class of 21 students to follow these activities for an 8-week period. A control classroom of 23 third graders followed the same curriculum without the activities. At the end of the 8 weeks, all students took a Degree of Reading Power (DRP) test, which measures the aspects of reading ability that the treatment is designed to improve.

(1) Suppose the distribution of scores on the DRP test are normally distributed with mean 49 and variance 4. What is the probability that a randomly selected student taking the exam will score above 53?

The test data is given in excel spreadsheet “ImprovingReading.” Using these data, answer the following questions.

(1) Is the average Treatment mean significantly higher than 48?
(2) Determine whether the new directed reading activities produced significantly higher results.