(1) Real 1

(a) Define what it means for a sequence $(a_n)_{n=1}^{\infty}$ to diverge to $+\infty$. We write this $\lim_{n\infty\infty} a_n = +\infty$.

(b) Prove that if (a_n) is non-decreasing and there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} a_{n_k} = +\infty$$

then

$$\lim_{n\infty\infty} a_n = +\infty.$$

Define

$$s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

so that s_n is the n-th partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

(c) Show that

$$s_9 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} > \frac{9}{10}.$$

(d) Generalize this to show that

$$s_{10^k - 1} > k\frac{9}{10}.$$

(e) Hence show that $\lim_{n\to\infty} s_n = +\infty$.

(2) Real 2

(a) Define what it means for f to be differentiable at a.

(b) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable at a. Show that

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

(c) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. What does the Fundamental Theorem of Calculus say about $\int_a^b f'$?

(d) Prove that if $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable then

$$\int_a^b f \cdot g' = f(b) \cdot g(b) - f(a) \cdot g(a) - \int_a^b f' \cdot g$$

(3) Discrete Probability

(a) Prove the *Binomial Theorem*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

(b) Use the Binomial Theorem to prove that for any $p \in \mathbb{R}$

$$1 = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

and

$$(1-2p)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k p^k (1-p)^{n-k}$$

(c) Hence prove that

$$\frac{1-(1-2p)^n}{2} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} p^k (1-p)^{n-k}.$$

(d) Consider the Binomial random variable of the number of successes from n independent Bernoulli trials with the probability of success p. Show that if n is even and $p \neq \frac{1}{2}$ then you are more likely to get an even number of successes than an odd number of successes.

(4) Linear Algebra

(a) Define the linear span of vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$.

(b) Define the row space of a matrix A.

(c) Show that the row space of a matrix is invariant under row operations.

(d) Use row reduction to find a basis for the subspace spanned by the vectors $(1, 2, 3, 4, 5), (6, 4, 3, 2, 0), (0, 2, 9, 10, 18), (1, 0, 1, 0, 1), (7, -4, -4, -14, -15)$.

(e) Explain the difference between using a row space and a column space when using row reduction of a matrix to find a basis for the span of a set of vectors.

(5) Number Theory

(a) Let a,b,and c be positive integers. Show that the Diophantine equation

$$ax + by = c$$

has solutions $x, y \in \mathbb{Z}$ if and only if $\gcd(a, b) | c$.

(b) Find a solution to the Diophantine equation

$$156x + 91y = 39.$$

(c) Find *all* solutions to the Diophantine equation

$$156x + 91y = 39.$$