QUALIFYING EXAM (FALL 2000)
ALGEBRA

Answer any **six** of the following eight questions. **You must state clearly any general results you use.**

1. Let G be a finite group and let p be the smallest prime factor of $|G|$. Show that any subgroup H of index p in G is normal in G.
2. Classify all groups of order $245 = 5.7^2$ up to isomorphism, stating clearly any results you use.
3. Let K be a field.
   (a) Show that $K[X]$ is a PID.
   (b) Show that $K[X, Y]$ is not a PID.
   (c) Explain why both $K[X]$ and $K[X, Y]$ are UFDs, stating clearly any results that you use.
4. Let R be a commutative ring with 1 and let I be an ideal of R.
   (a) Show that I is a maximal ideal if and only if R/I is a field.
   (b) Show that I is a prime ideal if and only if R/I is an Integral Domain.
   (c) Show that if I is maximal then I is prime.
   (d) Show that if R is a PID and I is a non-zero prime ideal then I is maximal.
5. Let F be a field and let $f(x)$ be an irreducible polynomial over F. Show that if K is a Galois extension of F then all the irreducible factors of $f(x)$ in $K[x]$ have the same degree.
6. Find the Galois group of $X^3 + X + 1$ over
   (a) $\mathbb{F}_2$ (the field of 2 elements),
   (b) $\mathbb{F}_3$ (the field of 3 elements),
   (a) $\mathbb{Q}$.

   State clearly any results you use.
7. Let R be a commutative ring with 1 with ideals I and J.
   (a) Prove $R/I \otimes_R R/J \cong R/(I + J)$.
   (b) Show that $R[X] \otimes_R R[X] \cong R[X]$ as R-modules. [Note: They are *not* isomorphic as R-algebras.]
8. Let A be an matrix over $\mathbb{C}$ such that $A^3 = A$. Show that A is similar to a diagonal matrix.