# Qualifying Exam (Spring 2003)
# Algebra

Answer any **five** of the following eight questions.
**You should state clearly any general results you use.**

1. Give an example of a non-cyclic proper subgroup of $(\mathbb{Q}, +)$.

2. Classify all groups of order $1225 = 5^2.7^2$ stating clearly any results that you use.

3. Let $R$ be a commutative ring with 1.

   (a) Show that every maximal ideal is prime.

   (b) Show that if $R$ is a PID then every non-zero prime ideal is maximal.

4. A *Boolean ring* is a ring in which $x^2 = x$ for all $x$. Let $R$ be a commutative Boolean ring (with 1).

   (a) Show that $2x = 0$ for all $x \in R$.

   (b) Show that every prime ideal of $R$ is maximal.

5. Prove that a finite extension $K/F$ is simple if and only if there are only finitely many intermediate fields.

6. What is the Galois group of $X^5 + 15X^2 - 70X + 15$ over

   (a) $\mathbb{F}_2$ (the field of 2 elements),

   (b) $\mathbb{F}_3$ (the field of 3 elements),

   (c) $\mathbb{Q}$.

   State clearly any results you use.

7. Suppose $A$ is a finitely generated abelian group and $A \oplus A \cong A$. Show that $A = 0$. Give an example of an abelian group $A \neq 0$ with $A \oplus A \cong A$.

8. Let $R$ be a commutative ring with 1 with ideals $I$ and $J$. Prove that $R/I \otimes_R R/J \cong R/(I + J)$.