Qualifying Exam: Algebra

Fall 2004

Answer any **five** of the following eight questions. Time allowed: **3 hours**.
You should state clearly any general results you use.

1. Let G be a finite group and let p be a prime. Let P be a Sylow p-subgroup of G. Let N_P be the normalizer of P (i.e. the set of $g \in G$ such that $gPg^{-1} = P$). Show that N_P is its own normalizer.

   [Hint: First show that P is the unique p-Sylow subgroup of N_P.]

2. Let p be a prime and let L be a linear map $\mathbb{F}_p^n \to \mathbb{F}_p^n$. Suppose that $L^{p^k} = 1$ for some $k \geq 0$, where 1 is the identity map on $\mathbb{F}_p^n$. Show that there exists a non-zero vector $v \in \mathbb{F}_p^n$ such that $L(v) = v$.

   [Hint: Consider the action of the group generated by L on $\mathbb{F}_p^n$.]

3. Let R be a commutative ring. An element a of R is called *nilpotent* if $a^n = 0$ for some $n \geq 1$. Show that the set of nilpotent elements of R forms an ideal.

4. Show that $\mathbb{Z}[X]$ is not a Principal Ideal Domain.

5. Let f be the polynomial $X^p - X - 1$ in the ring $\mathbb{F}_p[X]$. Let K be a splitting field for f and let α be a root of f in K. Let σ be the Frobenius automorphism of K, $\sigma(x) = x^p$.

   (a) Show that $\{\alpha, \alpha + 1, \ldots, \alpha + p - 1\}$ is the set of roots of f.

   (b) Show that $K = \mathbb{F}_p(\alpha)$.

   (c) What is the order of σ? (Hint consider $\sigma(\alpha)$).

   (d) Show that f is irreducible.

Please Turn Over

6. For each of the following, indicate if they are true or false, giving a brief justification for your answer.
   (a) There is a unique field with 36 elements.
   (b) If $[K : F] = n$ and f is an irreducible polynomial of degree m where $\gcd(m, n) = 1$, then f has no roots in K.
   (c) If L/K is a Galois extension and K/F is a Galois extension then L/F is a Galois extension.
   (d) In a principal ideal domain, any irreducible element is prime.

7. Suppose R is a ring and A and B are two R-modules. Let $f\colon A \to B$ and $g\colon B \to A$ be two R-linear maps with $g \circ f = 1_A$. Show that $B \cong \mathrm{Im}(f) \oplus \mathrm{Ker}(g)$.

8. Let A be a 3×3 matrix with complex entries and assume $A^2 = 0$, but $A \neq 0$. Give, with justification, the Jordan Normal Form of A.