PhD Qualifying Exam: Algebra

April 9, 2005

Answer any **five** *of the following eight questions.*
You should state clearly any general results you use.

1. Suppose G is a finite group and K is a normal subgroup of G with $\gcd(|K|, [G:K]) = 1$. Show that K is the unique subgroup of G of order $|K|$.

2. Show that there are no simple groups of order 30.

3. Let R be an integral domain. A non-zero non-unit element $s \in R$ is called special if for every $a \in R$ there exist $q, r \in R$ with $a = qs + r$ and such that either $r = 0$ or r is a unit in R.
   (a) If $s \in R$ is special, show that the principal ideal (s) is maximal.
   (b) Show that every polynomial of degree 1 in $\mathbb{Q}[X]$ is special in $\mathbb{Q}[X]$.
   (c) Prove that there are no special elements in $\mathbb{Z}[X]$.

4. Let F be a field and let $R = \{\sum_{i=0}^{n} a_i X^i : n \in \mathbb{N}, a_1 = 0\}$ be the subring of the polynomial ring $F[X]$ consisting of all polynomials with X-coefficient equal to 0.
   (a) Show that X^2 is irreducible but not prime in R.
   (b) Show that the ideal of R consisting of all polynomials in R with constant term 0 is not principal.

Turn over

5. Suppose $F \subseteq \mathbb{C}$ and $F/\mathbb{Q}$ is a finite Galois extension with $\mathrm{Gal}(F/\mathbb{Q})$ abelian. Let $\alpha \in F$ and assume $|\alpha| = 1$ where $|\alpha|$ is the absolute value of α considered as an element of $\mathbb{C}$.
   (a) Show that F is closed under complex conjugation. [Hint: $F/\mathbb{Q}$ is normal.]
   (b) If $m_\alpha(X) \in \mathbb{Q}[X]$ is the minimal polynomial of α over $\mathbb{Q}$ and β is another root of m_α, show that $|\beta| = 1$. [Hint: use (a) and $\mathrm{Gal}(F/\mathbb{Q})$ abelian.]
   (c) Writing $m_\alpha(X) = \sum_{i=0}^{n} a_i X^i$ show that $|a_i| \leq 2^n$.
   (d) Deduce that F contains only finitely many α with $|\alpha| = 1$ and $m_\alpha \in \mathbb{Z}[X]$, and each of these is a root of unity.

6. Find the Galois group of $X^4 - 5X^2 + 6$ over
   (a) $\mathbb{F}_3$ (the field with 3 elements),
   (b) $\mathbb{F}_5$ (the field with 5 elements),
   (c) $\mathbb{Q}$.

7. Let A be an 4×4 matrix with complex entries and suppose $A^3 = A^2$. List all the possible Jordan canonical forms for A, and in each case give both the minimal and characteristic polynomials of A.

8. Show that if A is a finite abelian group and $A \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z}) = 0$ for all primes p, then $A = 0$. Does this result remain true if A is infinite? Explain.