PhD Qualifying Exam: Algebra

September 9, 2006

Answer any **five** *of the following eight questions.*
You should state clearly any general results you use.

1. Let K be a normal subgroup of the finite group G. If G/K contains an element of order n, show that G contains an element of order n.

2. Let G be a finite group with $|G| = 420 = 2^2 \cdot 3 \cdot 5 \cdot 7$ and suppose that $P \leq G$ is a subgroup of order 7. Assume that $C_G(P) = P$, where $C_G(P) = \{g \in G : \forall h \in P \colon gh = hg\}$ is the centralizer of P.
   (a) If $K \trianglelefteq G$, show that either $P \leq K$ or $|K| \equiv 1 \bmod 7$.
   (b) Show that every non-trivial normal subgroup of G contains P.
   [Hint: If $P \not\leq K$, Consider 3- or 5-Sylow subgroups of G.]

3. Let R be a commutative ring with 1 and suppose that M is an ideal of R.
   (a) If M is both maximal and principal, show that there is no ideal I of R satisfying $M^2 < I < M$, where $<$ denotes strict inclusion.
   (b) Give examples to show that neither of the two conditions on M in part (a) can be removed.

4. Let $R = \mathbb{Z}[X]$ be the polynomial ring over the integers. Which of the following ideals are prime, and which are maximal.
   (a) $(2X)$
   (b) $(X^2 + 1)$
   (c) $(3, X^2 + 1)$
   (d) $(5, X^2 + 1)$

5. Find the Galois group of $f(X) = X^4 - 2$ over the fields
   (a) $\mathbb{F}_5$,
   (b) $\mathbb{R}$,
   (c) $\mathbb{Q}$.

6. Let F be a field and let $f(X) \in F[X]$ be an irreducible polynomial. Suppose K is an extension field of F containing an element α such that $f(\alpha) = f(\alpha^2) = 0$. Show that $f(X)$ splits over K.

7. Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices with complex entries. Describe all matrices $A \in M_n(\mathbb{C})$ with the property that all matrices of $M_n(\mathbb{C})$ that commute with A are diagonalizable.

8. Suppose $n, m > 0$ are integers. Show that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ where $d = \gcd(n, m)$.