Algebra Ph.D. Qualifying Exam

September 2007

Answer any **five** of the following eight questions.
You should state clearly any general results you use.

1. How many abelian groups are there of order 288 ($= 2^5 \times 3^2$) up to isomorphism?

2. Let $GL_n(F)$ be the group of invertible $n \times n$ matrices with entries in a field F under matrix multiplication.

   (a) Show that the center of $GL_n(F)$ is $\{\lambda I_n : \lambda \in F^\times\}$, where I_n is the identity matrix.

   (b) Show that $|SL_2(\mathbb{F}_3)| = 24$, where $\mathbb{F}_3$ is the field with 3 elements and $SL_2(\mathbb{F}_3)$ is the subgroup of matrices in $GL_2(\mathbb{F}_3)$ of determinant 1.

   (c) Deduce from (a) that $SL_2(\mathbb{F}_3)$ is **not** isomorphic to the symmetric group S_4.

3. Prove that $G =< a, b \mid b^2 = 1,\ ba^2b = a^3 >$ is the dihedral group of order 10.

4. Let R be a commutative ring.

   (a) Show that every maximal ideal of R is a prime ideal.

   (b) Give an example (with justification) of a ring R and a prime ideal of R that is not maximal.

5. (a) Show that every PID is a UFD.

   (b) Give an example (with justification) of a UFD that is not a PID.

6. Calculate the Galois group of $X^4 - 8X^2 + 15$ over the fields

   (a) $\mathbb{Q}$,

   (b) $\mathbb{F}_7$.

7. (a) Show that there exists a field extension K/F of degree 4 with no intermediate field L, $F \subsetneq L \subsetneq K$. (You may assume there exists Galois extensions with Galois group S_n for any n. Hint: A_4 has no subgroup of order 6.)

   (b) Show that if K/F is Galois of degree 4, then there must be an intermediate field L, $F \subsetneq L \subsetneq K$.

8. Show that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(n, m)$.