PhD Qualifying Exam: Algebra

February 2, 2008

Answer any **five** *of the following eight questions.*
You should state clearly any general results you use.

1. Let M and N be normal subgroups of G such that $G = MN$ show that $G/M \cap N$ is isomorphic to $G/M \times G/N$.

2. If G is a group of order $3825 = 3^2.5^2.17$ and H is a normal subgroup of G of order 17 then $H \leq Z(G)$. State all the major results you have used in your solution.

3. Choose one from the following three:
   (a) Every proper ideal in a ring with identity is contained in a maximal ideal.
   (b) In a commutative ring R, P is a prime ideal in R if and only if the quotient ring R/P is an integral domain.
   (c) In a commutative ring R, M is a maximal ideal if and only if the quotient ring R/M is a field.

4. Prove that the rings $\mathbb{Z}[X]$ and $\mathbb{Z}[X, Y]$ are not isomorphic.

5. Let p be a prime number and n an integer with $n > 0$.
   (a) Show that the splitting field of $X^{p^n} - X$ over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is a field with exactly p^n elements.
   (b) Deduce that there exists an irreducible polynomial of degree n in $\mathbb{F}_p[X]$.

6. Find the Galois group of $X^4 - 4$ over the following fields.
   (a) $\mathbb{F}_5$,
   (b) $\mathbb{R}$,
   (c) $\mathbb{Q}$.

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7. State the Fundamental Theorem of Finitely Generated Abelian Groups. Give the number of nonisomorphic abelian groups of order 576. Justify your answer.

8. (a) Define the tensor product of two $\mathbb{Z}$-modules.

   (b) Prove that $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[X] \cong \mathbb{R}[X]$.