PhD Qualifying Exam: Algebra

September 12, 2009

Answer any **five** *of the following* **eight** *questions.*
You should state clearly any general results you use.

1. Determine all non-isomorphic groups of order 15.

2. Let G be a group of order 3825 which contains a normal subgroup, H, of order 17. Show that H is a subgroup of the center of G.

3. An element x of a ring is *nilpotent* if $x^n = 0$ for some $n > 0$.
   (a) Let R be a commutative ring with 1. Show that if x and y are nilpotent elements of R then $x + y$ is nilpotent and the set of all nilpotent elements is an ideal in R.
   (b) Give an example to show that (a) may fail if R is not commutative.

4. Let $D = \{a + b\sqrt{17} : a, b, \in \mathbb{Z}\}$ and let $F = \mathbb{Q}(\sqrt{17})$ be the field of fractions of D.
   (a) Show that $X^2 + X - 4$ is irreducible in $D[X]$ but not in $F[X]$.
   (b) Show that D is not a unique factorization domain.

5. Is a regular 5-gon constructible by a straightedge and compass? Explain your answer.

6. Let K be a Galois extension of $\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong S_5$. Show that K is the splitting field of a polynomial of degree 5 over $\mathbb{Q}$. [Hint: Consider the fixed field of $S_4 \leq S_5$.]

Please Turn Over.

7. Let R be a ring with 1, and recall that R is naturally a (left) R-module with respect to left multiplication.

   (a) Prove that R is a division ring if and only if R is a simple R-module.

   (b) Prove that R is a division ring if and only if every nonzero R-module contains a submodule isomorphic to R.

8. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic as (left) $\mathbb{Q}$-modules.