PhD Qualifying Exam: Algebra

September 11, 2010

Answer any **five** *of the following* **eight** *questions.*
You should state clearly any general results you use.

1. Prove that the symmetric group S_n is a maximal subgroup of S_{n+1}, i.e., if $S_n \leq H \leq S_{n+1}$ then either $H = S_n$ or $H = S_{n+1}$. (Here we regard S_n as the subgroup of permutations in S_{n+1} that fix the element $n+1$.)

2. Let N be a normal subgroup of G and let C be a conjugacy class of G that is contained in N. Prove that if $[G : N] = p$ is prime, then either C is a conjugacy class of N or C is a union of p distinct conjugacy classes of N.

3. Let R be a commutative ring with 1.
   (a) Show that if M is a maximal ideal of R then M is a prime ideal of R.
   (b) Give an example of a non-zero prime ideal in a ring R that is not a maximal ideal.
   (c) Show that if R is finite then every prime ideal of R is a maximal ideal.

4. Let p be a prime and let R be the ring of all 2×2 matrices of the form
$$\begin{pmatrix} a & b \\ pb & a \end{pmatrix}$$
where $a, b \in \mathbb{Z}$. Prove that R is isomorphic to $\mathbb{Z}[\sqrt{p}]$.

5. Show that the extension $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$ is Galois, and describe its Galois group.

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6. An extension K of a field F of characteristic $p \neq 0$, is called *purely inseparable* extension if for each $\alpha \in K$ there is an integer t such that $\alpha^{p^t} \in F$. Show that every purely inseparable field extension is a normal extension.

7. Let $T \colon \mathbb{Z}^n \to \mathbb{Z}^n$ be a $\mathbb{Z}$-linear map whose matrix with respect to the standard basis of $\mathbb{Z}^n$ is M. If $\det M \neq 0$, show that $\mathbb{Z}^n/\mathrm{Im}(T)$ is a finite group of order $|\det M|$, where $\mathrm{Im}(T)$ is the image of the map T.

8. Let R be a commutative ring with 1 and let N and M be two R-modules. Prove that $N \otimes_R M \cong M \otimes_R N$.