Algebra Ph.D. Qualifying Exam

September 2011

Answer any **five** of the following eight questions.
You should state clearly any general results you use.

1. (a) Show that if G is a nonabelian finite group then $|Z(G)| \leq \frac{1}{4}|G|$.
   (b) Give an example of a finite group with $|Z(G)| = \frac{1}{4}|G|$.

2. Let G be a finite group acting on a set X of size n and suppose for any $a_1, a_2, b_1, b_2 \in X$ with $a_1 \neq a_2$ and $b_1 \neq b_2$, there exists a $g \in G$ such that $g \cdot a_i = b_i$ for $i = 1, 2$. Show that $|G|$ is divisible by $n(n-1)$. [Hint: consider the action of $\mathrm{Stab}_G(a)$ on $X \setminus \{a\}$.]

3. Let R be a ring with 1, and n a positive integer. If $M_n(R)$ denotes the ring of $n \times n$ matrices with entries in R, prove that $M_n(I)$ is an ideal of $M_n(R)$ whenever I is an ideal of R, and that every ideal of $M_n(R)$ is of this form.

4. (a) Let R be a PID. Show that if P_1 and P_2 are prime ideals with $P_1 \subsetneq P_2$ then $P_1 = (0)$.
   (b) Give an example of a commutative ring and prime ideals P_1, P_2, with $(0) \subsetneq P_1 \subsetneq P_2$.

5. (a) Find the minimal polynomial m_α over $\mathbb{Q}$ of $\alpha = \sqrt{2 + \sqrt{6}}$.
   (b) Determine the Galois group of the splitting field extension of m_α over $\mathbb{Q}$.

6. Suppose $K = F(\alpha)$ is a non-trivial Galois extension of F and assume there exists an element $\sigma \in \mathrm{Gal}(K/F)$ such that $\sigma(\alpha) = \alpha^{-1}$. Show that $[K : F]$ is even and $[F(\alpha + \alpha^{-1}) : F] = \frac{1}{2}[K : F]$.

7. Let R be an ID and M an R-module. Define the rank $\mathrm{rk}(M)$ of M to be the maximum size of a R-linearly independent subset of M. Prove that for $n \in \mathbb{N}$, $\mathrm{rk}(R^n) = n$, where R^n denotes a direct sum of n copies of R.

8. Let R be a subring of a commutative ring S and consider S as an R-module. If S is isomorphic (as a module) to a direct sum of n copies of R, show that S is isomorphic (as a ring) to a subring of $M_n(R)$, the ring of $n \times n$ matrices with entries in R.