Qualifying Exam of Algebra-Spring 2011

Do five of the following eight problems. You should state clearly any general results you use.

1. Let G be an *abelian group*. Let $K = \{a \in G : a^2 = 1\}$ and $H = \{a^2 : a \in G\}$. Show that $G/K = H$.

2. (a) Let S_n be the symmetric group on n elements. Suppose $\sigma \in S_n$ has order 21. Show that $n \geq 10$.

   (b) Let G be a finite simple group containing an element of order 21. Show that every proper subgroup H of G has index at least 10. [Hint: consider an action of G on the left cosets on H.]

3. Let $R = C[0,1]$ be the ring of all continuous real-valued functions on $[0,1]$, with addition and multiplication defined pointwise: $(f+g)(x) = f(x)+g(x)$, $(fg)(x) = f(x)g(x)$. Prove that if M is a maximal ideal of R, then there is a real number $x_0 \in [0,1]$ such that $M = \{f \in R : f(x_0) = 0\}$.

4. Let R be a commutative ring with 1 such that for every $x \in R$ there is an integer $n > 1$ (depending on x) such that $x^n = x$. Show that every prime ideal of R is maximal. [Hint: first show that R is ID then it is a field.]

5. Let R be an integral domain and F a subring of R which is a field. Show that if each element of R is algebraic over F, then R is a field.

6. Find the Galois group of $X^4 - 2$ over the fields.

   (a) $\mathbb{F}_5$,

   (b) $\mathbb{Q}(i)$.

7. Prove that an $n \times n$ matrix with entries in a field F is diagonalizable if and only if its minimal polynomial factors into distinct linear factors in in $F[X]$.

8. Suppose S is a commutative ring with 1 and R is a subring of S. Suppose also that M is an R-module. Show that $S \otimes_R M$ can be given an S-module structure. [You need not verify all the module axioms, but you should at least define scalar multiplication and prove that it is well-defined.]