Algebra Ph.D. Qualifying Exam

September 2012

Answer any **five** of the following eight questions.
You should state clearly any general results you use.

1. Let π be an element of the symmetric group S_n and let $\tau \in S_n$ be a transposition. Show that the number of cycles in the cycle decomposition of $\tau\pi$ is either one more or one less than the number of cycles in the cycle decomposition of π.

2. Let G be a finite group. Show that if G has a normal subgroup K of order 3 that is not contained in the center of G, then G has a subgroup of index 2. [Hint: The group G acts on K by conjugation.]

3. Let R be a principal ideal domain.
   (a) For $a, b \in R$, define a *least common multiple* of a and b.
   (b) Show that $d \in R$ is a least common multiple of a and b if and only if $(a) \cap (b) = (d)$.

4. (a) How many units does the ring $\mathbb{Z}/60\mathbb{Z}$ have? Explain your answer.
   (b) How many ideals does the ring $\mathbb{Z}/60\mathbb{Z}$ have? Explain your answer.

5. Show that the field $K = \mathbb{Q}(e^{2\pi i/5})$ does not contain $i = \sqrt{-1}$.

6. (a) Show that the Galois group of $X^6 - 2$ over $\mathbb{Q}$ is dihedral of order 12.
   (b) List all subfields of $\mathbb{Q}(\sqrt[6]{2})$, explaining clearly why your list is complete.

7. A complex matrix A has characteristic polynomial $(X-2)^5$ and minimal polynomial $(X-2)^3$. List all possible Jordan Normal Forms for A.

8. Let M, M', N, N' be R-modules and $f\colon M \to M'$ and $g\colon N \to N'$ R-linear maps. Show that there is a unique R-linear map $h\colon M \otimes N \to M' \otimes N'$ such that $h(m \otimes n) = f(m) \otimes g(n)$.