Algebra Ph.D. Qualifying Exam

January 2012

Answer any **five** of the following eight questions.
You should state clearly any general results you use.

1. The exponent $\exp(G)$ of a group G is the smallest $k \geq 1$ such that $g^k = 1$ for all $g \in G$, or ∞ if no such k exists.

   (a) Show that a finitely generated abelian group A with $\exp(A) < \infty$ is finite.

   (b) Give an example of an infinite group of finite exponent.

   (c) Give an example of a group G in which every element has finite order but $\exp(G) = \infty$.

2. Prove that any group of order 182 is solvable. (Note that $182 = 2 \cdot 7 \cdot 13$).

3. Let $i = \sqrt{-1} \in \mathbb{C}$ and let x be an indeterminate.

   (a) Show that the three *additive groups* $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}[i]$, and $\mathbb{Z}[x]/(x^2)$ are all isomorphic to each other.

   (b) Show that no two of the three rings $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}[i]$, and $\mathbb{Z}[x]/(x^2)$ are isomorphic to each other.

4. Let $R = \mathbb{R}[u, v]/(v^2 - u^3)$ where u and v are indeterminants. You may assume that R is an integral domain.

   (a) Show that $f(x) = X^2 - u$ has a root in the field of fractions of R, but not in R.

   (b) Deduce that R is not a unique factorization domain.

5. Let $f(X) = X^4 + 3X + 9$. For each of the following groups, either exhibit a prime p such that this group is isomorphic to the Galois group of f over $\mathbb{F}_p$, or explain why no such prime p exists.

   (a) C_4,

   (b) C_8,

   (c) $C_2 \times C_2$.

6. Let $\mathbb{C}(t) = \{\frac{p(t)}{q(t)} : p, q \in \mathbb{C}[t],\ q \neq 0\}$ be the field of rational functions in the indeterminant t. Suppose $f(t) \in \mathbb{C}(t)$ satisfies $f(t) = f(-1/t)$. Show that $f(t) = g(t - 1/t)$ for some $g(t) \in \mathbb{C}(t)$. [Hint: Let $\phi\colon \mathbb{C}(t) \to \mathbb{C}(t)$ be the automorphism that sends t to $-1/t$. What is the fixed field of $\mathbb{C}(t)$ under the group $\{1, \phi\}$?]

7. Let $T\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ be the linear map given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 3 & 4 \\ 1 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Identify the group $\mathbb{Z}^3/\operatorname{im} T$ up to isomorphism.

8. Show that any $n \times n$ complex matrix A can be written in the form $A = D + N$ where D is diagonalizable, N is nilpotent, and $DN = ND$. [Hint: for example $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.]