# Algebra Ph.D. Qualifying Exam

September 2013

Answer any **five** of the following eight questions.
**You should state clearly any general results you use.**

1. Classify all finite groups $G$ with the following property: for any $g, h \in G$, either $g$ is a power of $h$, or $h$ is a power of $g$.

2. Show that any group of order $616 = 2^3 \cdot 7 \cdot 11$ is solvable.

3. Let $R_1$ and $R_2$ be commutative rings.

   (a) Show that any ideal of $R_1 \times R_2$ is of the form $I_1 \times I_2$ where $I_1$ is an ideal of $R_1$ and $I_2$ is an ideal of $R_2$.

   (b) Show that any *prime* ideal of $R_1 \times R_2$ is either of the form $P_1 \times R_2$ or $R_1 \times P_2$ where $P_1$ is a prime ideal of $R_1$ and $P_2$ is a prime ideal of $R_2$.

4. Let $F$ be a field and let $F[X, X^{-1}]$ be the ring of "Laurent polynomials", i.e., all finite $F$-linear combinations $\sum_{i=-N}^{M} a_i X^i$ of integer powers of $X$, where negative powers are allowed. Show that $F[X, X^{-1}]$ is a PID.

5. (a) State the Tower law for finite field extensions.

   (b) Suppose $f$ and $g$ are two irreducible polynomials over the field $F$ and assume $\alpha$ is a root of $f$ in some extension field. If $\deg f$ and $\deg g$ are relatively prime, show that $g$ is irreducible in $F(\alpha)[X]$.

6. Let $f(X) = X^4 - 10X^2 + 21$. Find the Galois group of $f$ over the fields

   (a) $\mathbb{F}_3$,

   (b) $\mathbb{F}_5$,

   (c) $\mathbb{Q}$.

7. Let $A$ be a finitely generated Abelian group (with group operation written additively).

   (a) If $B$ is a subgroup of $A$ such that $A = B + pA$ for some prime $p$, show that $B$ is of finite index in $A$.

   (b) If $B$ is a subgroup of $A$ such that $A = B + pA$ for all primes $p$, show that $B = A$.

8. Suppose $A$ and $B$ are two $n \times n$ matrices with complex entries with the same minimal polynomials and the same characteristic polynomials.

   (a) If $n = 3$ show that $A$ and $B$ are similar.

   (b) Give an example of non-similar $A$ and $B$ with this property and $n > 3$.