Algebra Ph.D. Qualifying Exam

January 2013

Answer any **five** of the following eight questions.
You should state clearly any general results you use.

1. Let p be prime and let S_n be the symmetric group of all permutations of $\{1, \dots, n\}$.
   (a) If $n < 2p$, show that any two subgroups of S_n of order p are conjugate.
   (b) Show that (a) fails when $n \geq 2p$.
2. Show that for any prime p there are precisely two groups of order $2p$ up to isomorphism.
3. Recall that a ring R is *simple* if the only ideals of R are (0) and R.
   (a) Show that a commutative ring is simple iff it is a field.
   (b) Give an example **with proof** of a non-commutative simple ring that is not a division ring.
4. Let $\mathbb{Z}[i]$ be the ring of Gaussian integers.
   (a) Show that 3 is prime in $\mathbb{Z}[i]$ but 5 is not.
   (b) Show that, if a prime p in $\mathbb{Z}$ is not prime in $\mathbb{Z}[i]$, then either $p = 2$ or $p \equiv 1 \bmod 4$.
5. Let L/K and K/F be (possibly infinite) algebraic field extensions. Prove that L/F is algebraic.
6. Let $\alpha = \sqrt{5 + \sqrt{5}}$. Show that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois, and that its Galois group is cyclic.
7. Let $R = \mathbb{R}[X, Y]$ and let $I = (X, Y)$ be the ideal of R generated by X and Y.
   (a) Is I a free $\mathbb{R}[X, Y]$-module? Explain.
   (b) Is I a free $\mathbb{R}[X]$-module? Explain.
   (c) Is I a free $\mathbb{R}$-module? Explain.
8. Let A be an $n \times n$ complex matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ (counted with multiplicity). Let $f(X) \in \mathbb{C}[X]$ be a polynomial. Show that the determinant of $f(A)$ is $\prod f(\lambda_i)$.