NAME (print clearly):

PHD QUALIFYING EXAMINATION FOR ALGEBRA (FALL 2014)

Answer 5 of the following questions correctly.

Question 1. Determine the group defined by the following generators and relations:

$$\langle a, b : \ a^8 = e, \ b^2 a^4 = e, \ bab^{-1}a = e \rangle.$$

Question 2. Answer the following.

1. Define the composition series of a group G.
2. Determine the composition series of the group S_4.
3. Determine the composition series of the group S_5.

Question 3. Let $\theta : R \to S$ be an ring epimorphism from R into S with kernel K.

1. Prove each of the following:
   (i) If P is a prime ideal of R containing K, then $\theta(P)$ is a prime ideal of S.
   (ii) If Q is a prime ideal of S, then $\theta^{-1}(Q)$ is a prime ideal of R containing K.
   (iii) Prove that there is a one-to-one correspondence between the prime ideals of R containing K and the prime ideals of S.
2. If I is an ideal of R, determine the ideal $\theta^{-1}(\theta(I))$ of R.

Question 4. Let R be a commutative ring with identity and let $f(X) = \sum_{i=0}^{n} a_i X^i$ be a polynomial in $R[X]$. Prove that $f(X)$ is a unit in $R[X]$ if and only if a_0 is a unit and each a_i is nilpotent for $1 \leq i \leq n$.

Question 5. Let $f_1(X) = X^3 - 2X + 7$, $f_2(X) = X^4 + 3X^3 - 3X - 2$ and $f_3(X) = X^7 + 7X^3 + 6$ be polynomials in $\mathbb{Z}[X]$. Let α_i be a root of $f_i(X)$ for $i = 1, 2, 3$.

1. What is the degree of the field extension $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over $\mathbb{Q}$ (i.e, what is $[\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3), \mathbb{Q}]$ equal to)?
2. Let K_1 be the splitting field of $f_1(X)$ over $\mathbb{Q}$. Determine $[K_1 : \mathbb{Q}]$.
3. Determine the Galois group of $K_1/\mathbb{Q}$.
4. Is $\mathbb{Q}(\alpha_1)/\mathbb{Q}$ separable? Is it normal? Is it Galois?

For all parts, include a full justification for your answer. Simply writing down the correct group, number or response is not sufficient.

Question 6. Let $K/\mathbf{k}$ be a field extension of degree 2, so $[K : \mathbf{k}] = 2$.

1. If the characteristic of $\mathbf{k}$ is not 2, show that $K/\mathbf{k}$ is Galois.
2. Let $\mathbb{F}_2$ be the finite field of two elements and let $\mathbf{k} = \mathbb{F}_2(t)$ be a transcendental field extension of $\mathbb{F}_2$ (i.e., t is not algebraic over $\mathbb{F}_2$). Give an example of a field extension $K/\mathbf{k}$ such that $[K : \mathbf{k}] = 2$ but $K/\mathbf{k}$ is not Galois. Include all details showing why the example has the stated properties.

Question 7. Let R be a commutative ring with identity and let

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be a short exact sequence of R-modules.

1. Show that
$$0 \to R \otimes_R M' \xrightarrow{1 \otimes f} R \otimes_R M \xrightarrow{1 \otimes g} R \otimes_R M'' \to 0$$
is a short exact sequence of R-modules.
2. Let F be a free R-module. Show that
$$0 \to F \otimes_R M' \xrightarrow{1 \otimes f} F \otimes_R M \xrightarrow{1 \otimes g} F \otimes_R M'' \to 0$$
is a short exact sequence of R-modules.

Question 8. Let R be a commutative ring with identity, let $\mathbf{m} \subseteq R$ be a maximal ideal, and let $A = (R/\mathbf{m})[X]$ be the polynomial ring over $R/\mathbf{m}$. If M is a projective A-module, what can M be isomorphic to? Justify your answer.