Algebra Ph.D. Qualifying Exam

January 2014

Answer any **five** of the following eight questions.
You should state clearly any general results you use.

1. The exponent $\exp(G)$ of a group G is the smallest $k \in \{1, 2, \dots\} \cup \{\infty\}$ such that $g^k = 1$ for all $g \in G$.

   (a) Show that a finitely generated abelian group A with $\exp(A) < \infty$ is finite.

   (b) Give an example of an infinite group of finite exponent.

   (c) Give an example of a group G in which every element has finite order but $\exp(G) = \infty$.

2. Show that a group of order 80 cannot be simple.

3. Let R be a commutative ring with 1. Show that the sum of any two principal ideals of R is principal if and only if, every finitely generated ideal of R is principal.

4. (a) Show that $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\sqrt{3}]$, and $\mathbb{Z}[X]/(X^2)$ are isomorphic as additive groups. (Here X is an indeterminate.)

   (b) Show that $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\sqrt{3}]$, and $\mathbb{Z}[X]/(X^2)$ are *not* isomorphic as rings.

5. (a) Show that if K/F is a Galois extension and $[K : F]$ is a power of 2, then there exists intermediate fields $F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = K$ such that $[F_{i+1} : F_i] = 2$ for all $i = 0, \dots, n-1$.

   (b) Show that this need not be true if K/F is not Galois. [Hint: let K and F be suitable intermediate fields in a Galois extension $M/\mathbb{Q}$ with Galois group S_4. You may assume such an M exists.]

6. Let $f(X) = X^5 - 11$. Find the degree of the splitting field of f over the following fields.

   (a) $\mathbb{F}_2$.

   (b) $\mathbb{Q}(\zeta_5)$, where ζ_5 is a primitive 5th root of 1.

   (c) $\mathbb{Q}$.

7. Suppose T is a linear operator on an n-dimensional vector space V over a field F such that for any non-zero $v \in V$ the set $\{T^i(v) : i = 0, \dots, n-1\}$ linearly independent. Show that the characteristic polynomial of T is irreducible over F.

8. Identify with proof $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$.