ALGEBRA PH.D. QUALIFYING EXAM
SUMMER 2016

Answer FIVE of the following eight questions correctly (there are questions on the reverse side). When using a theorem from class or your text, be sure to cite the hypotheses needed to apply it and state what the resulting conclusion is. When giving an example, be sure to give a full justification as to why the example works.

All rings have identity.

Question 1. Let G_α for $\alpha \in I$ be groups and let $\prod_{\alpha\in I} G_\alpha$ be their direct product. Show that $Z(\prod_{\alpha\in I} G_\alpha) = \prod_{\alpha\in I} Z(G_\alpha)$, where $Z(G)$ denotes the center of the group G.

Question 2. Let G be a group. We say a proper subgroup $H < G$ is *maximal* if the only subgroups of G containing H are H and G. Let $\Phi(G)$ be the intersection of all maximal subgroups of G (set $\Phi(G) = G$ if G has no maximal subgroups).

1. Show that $\varphi(\Phi(G)) = \Phi(G)$ for any automorphism $\varphi : G \to G$.
2. Show that $\Phi(G) \trianglelefteq G$.

Question 3. Let R be the set of all continuous functions $f : [0, 1] \to \mathbb{R}$.

1. Show that R is ring with multiplication and addition of functions f, $g \in R$ given by $(fg)(x) = f(x)g(x)$ and $(f + g)(x) = f(x) + g(x)$, for $x \in [0, 1]$.
2. Let $I \subseteq R$ be those functions $f \in R$ with $f(1/2) = f(1/3) = 0$. Show that I is an ideal of R but not a prime ideal.

Question 4. Let F be a field, let $R = F[X]$ be the polynomial ring, let $R' \subseteq R$ be the subset of all polynomials whose X-coefficient is zero, and let $J \subseteq R'$ be the subset of all polynomials whose X-coefficient *and* constant term are both zero.

1. Show that R' is a ring.
2. Show that J is an R-ideal *and* also an R'-ideal.
3. Show that R' is not a principal ideal domain.
4. Show that any ideal $I \subseteq R'$ can be generated by at most two elements, so $I = R'g_1 + R'g_2$ for some g_1, $g_2 \in I$. Hint: use polynomial division and consider a nonzero element $g_1 \in I$ with $\deg g_1$ minimal.

Question 5. Let R be a ring and let $e \in R$ be an element such that $er = re$ for all $r \in R$. Let M be an R-module.

1. Show that $eM = \{ex : x \in M\}$ is a submodule of M.
2. If $e^2 = e$, show that $M = eM \oplus (1 - e)M$.

Question 6. Let R be a commutative ring, let $\mathfrak{p} \subseteq R$ be a prime ideal and let M and N be R-modules. Show that $(M \oplus N)_\mathfrak{p} \cong M_\mathfrak{p} \oplus N_\mathfrak{p}$.

TURN OVER

Question 7. Let $K/\mathbb{Q}$ be a normal field extension of degree $[K : \mathbb{Q}] = 26656 = 2^5 \cdot 7^2 \cdot 17$. Show that there is a unique intermediary field E with $[K : E] = 49$.

Question 8. What is the Galois group of the polynomial $f(X) = X^{75} - 1$ over $\mathbb{Q}$ isomorphic to? What is the order of this Galois Group?