NAME (print clearly):

PHD QUALIFYING EXAMINATION FOR ALGEBRA (WINTER 2022)

Answer FIVE of the following questions correctly. All answers should be justified with a counter-example or proof. When using a theorem from class, be sure to include the hypotheses needed to use it and what the resulting conclusion is. All rings have an identity.

Question 1. Let G be a group and let $H \triangleleft G$ be a normal, abelian subgroup whose centralizer $Z_G(H)$ in G has $|Z_G(H) : H| = 77$. Show that $Z_G(H)$ is abelian.

Question 2. Let H and K be groups with $|H| = 7^5$ and $|K| = 5^5$, and let $G = H \times K$. If $L \leq G$ is a subgroup with $|G : L| = 7$, show that L is normal in G.

Question 3. Let R be the ring of all continuous functions $f : [0, 1] \to \mathbb{R}$, equipped with pointwise multiplication and addition. For $c \in [0, 1]$, show that $M_c = \{f \in R : \ f(c) = 0\}$ is a maximal ideal.

Question 4. Let R be a Noetherian Domain. Show that, if a nonzero, prime ideal $P \subseteq R$ is contained in a proper, principal ideal $(z) \subsetneq R$, where $z \in R$, then $P = (z)$.

Question 5. Show that $\mathbb{C} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = 0$ but $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$.

Question 6. Show that the exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

is not split exact.

Question 7. In each of the following cases, for the field extension $F(\sqrt{5}, \sqrt{7})/F$, determine if the extension is Galois and, if so, determine its Galois group:

1. $F = \mathbb{Q}$
2. $F = \mathbb{F}_{17}$, the field with 17 elements.
3. $F = \mathbb{F}_2$, the field with two elements.

Question 8. Let K/k be a finite galois field extension with Galois Group G. If p is a prime with $p \mid |G|$, show that there is an intermediary field E, so $k \subseteq E \subseteq K$, with $|K : E| = p$.