ALGEBRA PH.D. QUALIFYING EXAM
SUMMER 2022

Answer FIVE of the following eight questions correctly. When using a theorem from class, be sure to cite the hypotheses needed to apply it and state what the resulting conclusion is. When giving an example, be sure to give a full justification as to why the example works.

All rings have identity.

Question 1. Consider the action of the symmetric group S_4 on the set of polynomials $\mathbb{Z}[X_1, X_2, X_3, X_4]$ defined by permuting the indices of each polynomial, i.e.,

$$f^\sigma(X_1, X_2, X_3, X_4) = f(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}, X_{\sigma(4)}), \qquad \text{for } \sigma \in S_4 \text{ and } f \in \mathbb{Z}[X_1, X_2, X_3, X_4].$$

(a) Explicitly describe the elements in the orbit of the polynomial $(X_1 + X_2)(X_3 + X_4)$.

(b) Explicitly describe the elements in the stabilizer of the polynomial $(X_1 + X_2)(X_3 + X_4)$.

Justify your answers!

Question 2. Show there exists a non-abelian group of order 55.

Question 3. Let $p \in \mathbb{Z}$ be a prime with $p \equiv 1 \mod 4$. Show that $\mathbb{Z}[\sqrt{p}] = \{a + b\sqrt{p} : \ a, b \in \mathbb{Z}\}$ is not a unique factorization domain.

Question 4. Let R be a commutative ring and let $I \subset R$ be a proper ideal. Show that there exists a prime ideal $P \subset R$ that contains I which is minimal (by inclusion) among all prime ideals containing I.

Question 5. Let R be a commutative ring. Show that $\mathsf{Hom}_R(R, R) \cong R$ (both as R-modules and rings).

Question 6. Let

$$0 \to \mathbb{Q}/\mathbb{Z} \to M \to M'' \to 0$$

be a short exact sequence of $\mathbb{Z}$-modules. Show there exists a short exact sequence of $\mathbb{Z}$-modules

$$0 \to M'' \to M \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Question 7. Let

$$f(X) = X^{11} + 15X^{10} + 45X^9 + 75X^2 \in \mathbb{Q}[X] \qquad \text{and}$$
$$g(X) = X^{10} + 14X^9 + 30X^8 - 45x^7 + 75X - 75 \in \mathbb{Q}[X].$$

Let α be a root of $f(X)$ and let β be a root of $g(X)$. What are the possibilities for $|\mathbb{Q}(\alpha) : \mathbb{Q}|$? What are the possibilities for $|\mathbb{Q}(\beta) : \mathbb{Q}|$? Justify your answers!

Question 8. Let $K/\mathbf{k}$ be a Galois field extension with $|K : \mathbf{k}| = 95$, and let E be an intermediary field. Show that $E/\mathbf{k}$ is normal.