Algebra Qualifying Exam

Winter 2023

Answer FIVE of the following eight questions correctly. When using a theorem from class, be sure to cite the hypotheses needed to apply it and state what the resulting conclusion is. When giving an example, be sure to give a full justification as to why the example works. All rings have identity.

1. Give an example (with proof) of a group G and subgroup $H \leq G$ such that the normalizer $N_G(H)$ of H is *not* normal in G.

2. Show that a group of order $3^4 \cdot 5^{20}$ is solvable. (partial progress was sufficient)

3. Consider the ideal $I = (2x, y^3)$ in the ring $\mathbb{Z}[x, y]$. Determine (with proof) all prime ideals $P \subseteq \mathbb{Z}[x, y]$ such that $I \subseteq P$ but there does not exist another prime ideal P' with $I \subseteq P' \subset P$.

4. Let $R = \mathbb{Z} + x\mathbb{Q}[x] \subseteq \mathbb{Q}[x]$ be the ring of polynomials with rational coefficients whose constant term is an integer. Show that the element $x \in R$ cannot be written as a finite product of irreducible elements from R.

5. Is $\mathbb{Z}/4\mathbb{Z}$ a $\mathbb{Z}/2\mathbb{Z}$-module? If yes, identify with a proof $\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})[x]$.

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6. Let $R = \mathbb{Z}[i]$ be the ring of Gaussian integers. Let $R' = \mathsf{Hom}_R\big(R/(2), R/(3+3i)\big)$ be the R-module of R-module homomorphisms between the quotient modules $R/(2)$ and $R/(3+3i)$, where $i = \sqrt{-1}$. Prove that R' is isomorphic to the R-module $R/(1+i)$.

7. A polynomial with integer coefficients can be interpreted over any field. Give an example (with proof) of three fields F_1, F_2 and F_3, and a single polynomial f with integer coefficients, such that the three Galois groups of f, over the fields F_1, F_2 and F_3, are all non-isomorphic. Recall here that the Galois group of a separable polynomial $f \in F[x]$ is the Galois group of the splitting field K for f, so $\mathsf{Gal}(K/F)$.

8. If K is a finite field extension of $\mathbb{R}$, then what are the possibilities for $|K : \mathbb{R}|$? Justify your answer with a proof.