Real Analysis Qualifying Exam

Spring 2003

Answer any five questions; Credit will be given for the best five questions
Show all working; State clearly all theorems that you apply

1. Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on $[0,1]$ with $\|f_n\|_1 = 1$. Suppose that for all $\epsilon > 0$, there exists an N such that for $j, k \geq N$, $m(\{x\colon |f_j(x) - f_k(x)| > \epsilon\}) < \epsilon$,
   (a) Prove that there exists a subsequence $f_{n_j}(x)$ such that for almost every $x \in [0,1]$, $\big(f_{n_j}(x)\big)_{j=1}^{\infty}$ is a convergent sequence.
   (b) Is there necessarily a subsequence f_{m_j} that is convergent in $L^1([0,1])$?

2. Show that if $f, g \in L^1(\mathbb{R})$, then their convolution $f * g$ defined by
$$f * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\,ds$$
is integrable.

3. Let A be a subset of $\mathbb{R}$ such that $m(A) > 0$ where m denotes Lebesgue measure. Denote by $A - A$ the set $\{x - y\colon x, y \in A\}$.
   (a) Prove that there is a bounded interval $[a,b]$ such that $m(A \cap [a,b]) > 3(b-a)/4$.
   (b) Show that if $0 \leq \delta \leq (b-a)/4$ then $A \cap (A + \delta) \cap [a,b]$ is non-empty.
   (c) Deduce that $A - A \supset [-(b-a)/4, (b-a)/4]$.

4. (a) State Fatou's lemma.
   (b) Give an example showing that the inequality in Fatou's lemma may be strict.
   (c) Starting from the monotone convergence theorem, give a proof of Fatou's lemma.

5. Consider the space $C([0,1])$ with the uniform metric $d(f,g) = \max_{t\in[0,1]} |f(t) - g(t)|$. You may assume that this space is complete. A continuous function on $[0,1]$ is called Lipschitz with constant A if $|f(x) - f(y)| \leq A|x - y|$ for all $x, y \in [0,1]$. A continuous function is called a Lipschitz function if it is Lipschitz with some constant A.

   (a) Show that for any $N \in \mathbb{N}$, the set S_N of continuous functions for which there exist rationals q_1 and q_2 such that $|f(q_1) - f(q_2)| > N|q_1 - q_2|$ is open and dense in the $C([0,1])$.

   (b) Deduce that the Lipschitz functions form a meager subset of $C[0,1]$.

6. Let $f \in L^1(\mathbb{R})$ satisfy $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and let $g \in L^\infty(\mathbb{R})$ satisfy $\lim_{x\to\infty} g(x) = L$.

   Prove that
$$\lim_{x\to\infty} \int_{-\infty}^{\infty} f(y)g(x-y)\,dy = L.$$

   [Hint: It may be helpful to prove that there is an $A \in \mathbb{R}$ such that $\int_{-\infty}^{A} |f|(x) < \epsilon$.]

7. Let $X = L^2([0,1])$ with the usual norm and let $Y = L^2([0,1])$ with the L^1 norm: $\|f - g\|_Y = \int_0^1 |f(x) - g(x)|\,dx$. Let $T\colon X \to Y$ be the identity map.

   Prove that if B denotes the open unit ball in X, then $T(B)$ is not open in Y.

   Explain **using the above** why it follows that Y is not a Banach space.