Real Analysis Qualifying Exam

Summer 2003

Answer any four questions; Credit will be given for the best four questions
Show all working; State clearly all theorems that you apply

1. Suppose that f and $f_1, f_2, f_3, \ldots$ are measurable functions on $\mathbb{R}$.
   (a) State what it means to say that the sequence (f_n) converges to f in measure.
   (b) Prove that if (f_n) converges to f in measure, then there is a subsequence of the (f_n) that converges pointwise to f almost everywhere.

2. Prove that if $1 \leq p, q < \infty$ and $p \neq q$, then any open ball in $L^p(\mathbb{R})$ (a set of the form $\{g \colon \|f - g\|_p < r\}$ for some $r > 0$) contains a function that does not belong to $L^q(\mathbb{R})$.

3. The first Borel-Cantelli Lemma states that if (X, μ) is a measure space and the sets $B_1, B_2, \ldots$ are measurable satisfying $\sum_{n=1}^{\infty} \mu(B_i) < \infty$, then the set of points that belong to infinitely many B_i is a set of measure 0.

   Prove the first Borel-Cantelli Lemma.

4. Consider the set S of L^1 functions that are essentially unbounded (a function is essentially unbounded if for each $C > 0$, there exists a set of positive measure on which $|f(x)| > C$). Prove that the set of essentially unbounded functions forms a dense G_δ set in L^1.

5. Let H be a Hilbert space and $g_1, g_2, \ldots$ be an orthonormal sequence. Prove that for any $f \in H$,
$$\sum_{i=1}^{\infty} |\langle f, g_i \rangle|^2 \leq \langle f, f \rangle$$
   with equality if and only if f is of the form $\sum_{i=1}^{\infty} a_i g_i$ for a square-summable sequence a_i.

6. For each part, prove the statement or give a counterexample.

(a) If (X, μ) is a measure space, then for any nested sequence of sets $A_1 \supseteq A_2 \supseteq \ldots$, $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

(b) If (X, μ) is a measure space, then for any nested sequence of sets $A_1 \subseteq A_2 \subseteq \ldots$, $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

(c) If (X, μ) is a measure space and f_n is a sequence of measurable functions, then $\limsup f_n$ is a measurable function.