PH.D. QUALIFYING EXAM
REAL VARIABLES, FALL 2004

You have three hours. Solve any five problems. Credit will be given for the best five questions. Show work.

1. A function f is said to satisfy Lipschitz condition on an interval $[a, b]$, if there is a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$.

   (a) Show that a function satisfying a Lipschitz condition is absolutely continuous.

   (b) Show that an absolutely continuous function f satisfies a Lipschitz condition if and only if $|f'|$ is bounded.

2. (a) Prove that a function of bounded variation on a closed interval is Lebesgue measurable.

   (b) Show that the product of two functions of bounded variation is a function of bounded variation.

3. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^p(X, \mathcal{B}, \mu)$, $1 \leq p < \infty$, where $(X, \mathcal{B}, \mu)$ is a σ-finite measure space.

   (a) Show that $\{f_n\}_{n=1}^{\infty}$ converges in measure μ on X.

   (b) Prove that $\{f_n\}_{n=1}^{\infty}$ contains a subsequence which converges almost everywhere in X.

4. State the Hahn-Banach theorem. Show that for any $x \in X$, where X is a Banach space, there exists a bounded linear functional f on X such that $f(x) = \|f\| \cdot \|x\|$.

5. Let μ, ν and λ be σ-finite. Show that the Radon-Nikodym derivatives $\left[\frac{d\nu}{d\mu}\right]$ and $\left[\frac{d\mu}{d\lambda}\right]$ have the following properties:

   (a) If $\nu \ll \mu$ and f is a nonnegative measurable function, then
   $$\int f \, d\nu = \int f \left[\frac{d\nu}{d\mu}\right] d\,\mu.$$

   (b) If $\nu \ll \mu \ll \lambda$, then
   $$\left[\frac{d\nu}{d\lambda}\right] = \left[\frac{d\nu}{d\mu}\right] \left[\frac{d\mu}{d\lambda}\right].$$

6. Let $f \in L^1(0, 1)$ and suppose that $\lim_{x \to 1^-} f(x) = A < \infty$. Prove that

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\,dx = A.$$

7. (*a*) If $f(x)$ is a real-valued Lebesgue-measurable function on $\mathbb{R}$, show that $F(x,y) = f(x-y)$ is a measurable function on $\mathbb{R}^2$.
   (*b*) If f and g are real-valued integrable functions on $\mathbb{R}$, show that for (Lebesgue) almost-all x, the function $\phi_x(y) = f(x-y)g(y)$ is integrable with respect to $y \in \mathbb{R}$.
   (*c*) Denote the Lebesgue integral of ϕ_x by $h(x)$. Show that h is integrable, and $\int_{\mathbb{R}} |h| \le \int_{\mathbb{R}} |f| \cdot \int_{\mathbb{R}} |g|$.