PhD Qualifying Exam: Real Analysis

March 13th 2004

Solve **five** *out of the seven problems.*
Show work. The exam lasts three hours.

1. Let S be a dense set of real numbers $\mathbb{R}$. Show that if f is an extended real-valued function defined on $\mathbb{R}$, such that $\{x : f(x) < \alpha\}$ is measurable for every $\alpha \in S$, then f is measurable.

2. Prove that $f(x) = \frac{\sin x}{x}$ is Riemann integrable but not Lebesgue integrable on $[1, \infty)$.

3. Define the sequence space ℓ^p as $\ell^p = \{<a_n> : \sum_n |a_n|^p < \infty\}$, where $1 < p < \infty$, and the norm on ℓ^p is given by by $\|<a_n>\|_p = \left(\sum_n |a_n|^p\right)^{1/p}$. Let F be a linear and bounded functional defined on ℓ^p. Prove that there is a unique sequence $<b_n> \in \ell^q$, where p, q are related by $\frac{1}{p} + \frac{1}{q} = 1$, such that $F(<a_n>) = \sum_n a_n b_n$. What is $\|F\|$? Prove your claim.

4. If $E \subset \mathbb{R}$ is a Lebesgue measurable set with a finite measure, prove that for any given $\epsilon > 0$, there is a set U which is a finite union of intervals such that $m(U \triangle E) < \epsilon$. Here $U \triangle E = (U \sim E) \cup (E \sim U)$, where $A \sim B = \{x \in A : x \notin B\}$.

5. If $f, g \in L^1[0,1]$ are positive functions on $[0,1]$ and $f(x)g(x) \geq 1$ for all $x \in [0,1]$, then
$$\left(\int_0^1 f\right)\left(\int_0^1 g\right) \geq 1.$$

Turn over

6. Let $C^\infty[0,1]$ denote the space of all functions $f : [0,1] \to \mathbb{R}$ continuously differentiable of all degrees. Let $C^\infty[0,1]$ be equipped with the norm

$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)|.$$

Define the linear operator $A\colon C^\infty[0,1] \to C^\infty[0,1]$ as

$$Af(x) = f'(x), \qquad x \in [0,1].$$

Show that A is unbounded but it has a closed graph. Is $C^\infty[0,1]$ under the norm $\|\cdot\|_\infty$, a Banach space?

7. State and prove the Lebesgue (Dominated) Convergence Theorem.