PhD Qualifying Exam: Analysis

September 3, 2005

Answer any **five** *of the following seven questions.*
You should state clearly any general results you use.

1. (a) State and prove the (Lebesgue) Dominated Convergence Theorem, stating clearly any results that you use.

   (b) Show that

$$\lim_{n\to\infty} \int_0^1 n^2 x(1-x)^n \, dx \neq \int_0^1 \lim_{n\to\infty} n^2 x(1-x)^n \, dx.$$

2. Let $f(x) = \frac{d}{dx}(x^2 \sin \frac{\pi}{x^2})$. Show that

$$\lim_{t\to 0} \int_t^1 f(x) \, dx$$

   exists, but that f is not integrable over $[0,1]$.

3. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a non-negative integrable function. Show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_S f(x)\, dx < \varepsilon$ for any Lebesgue measurable set S with Lebesgue measure $< \delta$.

4. Let $C([0,1])$ be the space of all continuous functions and $B([0,1])$ be the space of all bounded functions $f\colon [0,1] \to \mathbb{R}$. If we define a norm by $\|f\| = \sup_{x\in[0,1]} |f(x)|$ on both spaces, show that $C([0,1])$ is separable, but $B([0,1])$ is not.

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5. Let X and Y be normed linear spaces and suppose A is a subset of X such that the linear span of A is dense in X. If $T_n\colon X \to Y$ is a sequence of bounded linear functions such that

   (i) $\sup_n \|T_n\| < \infty$; and

   (ii) $T_n(a) \to 0$ for all $a \in A$,

   show that $T_n(x) \to 0$ for all $x \in X$.

6. Suppose f_n and f are measurable real-valued functions.

   (a) Define what it means to say that $f_n \to f$ in measure.

   (b) Show that if $f_n \to f$ in measure then there is a subsequence f_{n_k} that converges to f a.e..

   (c) Give an example of functions f_n and f such that $f_n \to f$ in measure, but $f_n(x) \not\to f(x)$ for all x.

7. Suppose $f \in L^1(\mathbb{R})$ and define $g(x) = xf(x)$. Show that

$$\|f\|_1^2 \leq 8\|f\|_2\|g\|_2.$$

   [Hint: $\int |f| = \int_{|x|<c} 1.|f| + \int_{|x|\geq c} \frac{1}{|x|}.|g|$. Apply Hölder.]