Real Analysis Ph.D. Qualifying Exam

April 2, 2005

Instructions. *Solve **four** of the six problems. Show your work. The exam lasts for three hours.*

1. Let $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable, then the function h, defined by

$$h(x) = H(f(x), g(x))$$

for $x \in \mathbb{R}$ is Lebesgue measurable.

2. Let $f : \mathbb{R} \times (a, b) \to \mathbb{R}$ and assume that $f(\cdot, y)$ is Lebesgue integrable on $\mathbb{R}$ for each $y \in (a, b)$. Define

$$g(y) = \int_{-\infty}^{+\infty} f(x, y)\ dx.$$

(a) Suppose that

$$\left| \frac{\partial f}{\partial y}(x, y) \right| \leq h(x)$$

for all $y \in (a, b)$ and $x \in \mathbb{R}$ where h is an integrable function on $\mathbb{R}$. Show that

$$g'(y) = \int_{-\infty}^{+\infty} \frac{\partial f}{\partial y}(x, y)\ dy.$$

(b) Find $g'(y)$ if

$$g(y) = \int_0^\infty e^{-xy} \frac{\sin x}{x}\ dx$$

for $y > 0$. Hint: Do it for $0 < \delta < y$.

3. Let $L^p[0,1]$ be the set of $f : [0,1] \to \mathbb{R}$ such that

$$\|f\|_p = \left(\int_0^1 |f(x)|^p\ dx \right)^{\frac{1}{p}} < \infty,$$

where $1 \le p < \infty$.

(a) Prove that $\|\cdot\|_p$is a norm.

(b) Prove that $L^p[0,1]$ is complete under the norm $\|\cdot\|_p$.

(c) Prove that $L^p[0,1]$ is separable under the norm $\|\cdot\|_p$.

4. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $f : X \to \mathbb{R}$ be a $\mu-$integrable function. Show that its support, that is, the set

$$\mathcal{S} = \{x \in X : f(x) \neq 0\}$$

is measurable and has σ-finite measure.

5. Let K be a subset of $\mathbb{R}^n$, $C(K)$ the continuous real-valued functions on K and $\mathcal{F}$ a set of functions defined on K.

   (a) Define " $\mathcal{F}$ is equicontinuous on K".

   (b) State (for compact K) the Arzela-Ascoli Theorem giving a criterion for the compactness of $\mathcal{F} \subseteq C(K)$.

   (c) Let $\{f_n\}$ be an equicontinuous sequence of functions from $[0,1]$ into $\mathbb{R}$ which converges at every rational number in $[0,1]$. Show that $\{f_n\}$ converges at every point of $[0,1]$ and the convergence is uniform.

6. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0 \\ \\ 0 & \text{for x} \leq 0 \end{cases} .$$

   Set

$$g(x) = f(x) f(1-x) .$$

   Show that g is a nontrivial infinitely differentiable function on $\mathbb{R}$ which vanishes outside $(0,1)$.

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